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The Influence of the Computing Precision on Interval Computations

Nathalie Revol

INRIA, Pascaline team - LIP, ENS de Lyon - France

Nathalie.Revol@ens-lyon.fr

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Introduction

For numerical computations, the use of the minimal useful computing precision enables one to reduce both the computing time and the memory usage. Indeed, most processors offer single precision and double precision (floating-point numbers stored on 32 bits, resp. on 64 bits) and each operation on 32-bit operands is twice faster (or even faster) than the corresponding operation on 64-bit operands. The energy consumption, memory bandwidth as well as storage requirements follow the same trend. However, users do not want to sacrifice the numerical accuracy of the final results. The question is thus to choose the adequate computing precision for each part of the algorithm, in order to improve the performances without significantly degrading the numerical quality [4, 1]. Such algorithms are said to use "mixed-precision".

Interval arithmetic is a tool to obtain numerical guarantees, by providing enclosures of the computed results [7, 6, 12]. It can also benefit from a sound choice of the computing precision: we will survey the use of mixed-precision in interval arithmetic for matrix multiplication, certification of the solution of a linear system, and Newton method.

Intervals can be represented by their endpoints: $\mathbf{x} = [\underline{x}, \bar{x}] = \{x : \underline{x} \leq x \leq \bar{x}\}$ and the endpoints of the interval $\mathbf{x}$ are $\underline{x}$ and $\bar{x}$, or by the *mid-rad* representation, using the midpoint and the radius of the interval: $\mathbf{x} = \langle x_m, x_r \rangle$ with $x_r \geq 0$ is the interval $[x_m - x_r, x_m + x_r]$.

Matrix multiplication

In [11], Rump proposes a fast algorithm for interval matrix multiplication. It uses the mid-rad representation of the two matrices to be multiplied: $\mathbf{A} = \langle A_m, A_r \rangle$ and $\mathbf{B} = \langle B_m, B_r \rangle$, and it trades off accuracy for speed by computing a matrix $\mathbf{C} = \langle C_m, C_r \rangle$ such that $\mathbf{C} \supset \mathbf{A} \times \mathbf{B}$, but the overestimation can be up to 50%, that is, for each coefficient of $\mathbf{C}$, its radius can be 1.5 times the radius of the corresponding coefficient of $\mathbf{A} \times \mathbf{B}$.

It uses 3 floating-point matrix multiplications, which are highly optimized routines in many libraries, usually called `gemm`, and it is much faster than the use of the classical formulae.

Then several authors [8, 10] have proposed other algorithms that either use more floating-point matrix multiplications, that is, more calls to `gemm`, in order to reduce the overestimation, or less calls to `gemm` in order to reduce the computing time.

Some of these variants have been studied by Ozaki et al. [9] in order to reduce the computing precision during the computation of the radius matrix. Indeed, in the case of narrow intervals, the radius conveys only the error or the inaccuracy on the midpoint, and there is no use in keeping many digits for the radius. Thus Ozaki et al. suggest to perform all computations related to the radii in small precision. Their main contribution is to account for the roundoff errors incurred by this reduction of precision, in order to preserve the inclusion property.

Certified solutions of linear systems

Another direction of research is the certification of the solution of a (either floating-point, or extremely tiny) linear system. The floating-point linear system is $Ax = b$, with A and b are of dimensions respectively $n \times n$ and n and have floating-point coefficients. We assume that the linear system is solved using a LU factorization of A and that $\hat{x}$ is an approximate solution. Let us denote by x^* the exact solution, that is, $Ax^* = b$ in $\mathbb{R}^n$. An iterative refinement approach is used.

In [8], the proposed algorithm uses a reduced computing preci-

sion for every time-consuming parts, namely the LU factorization and then the linear system solving using LU (which amounts to solve two linear triangular systems), that require respectively $\mathcal{O}(n^3)$ and $\mathcal{O}(n^2)$ floating-point operations. This algorithm uses the largest computing precision only for the numerically sensitive parts, namely the computation of the residual $r = b - A\hat{x}$ and the correction of $\hat{x}$ as $\hat{x} + \text{mid}(\mathbf{e})$, where $\mathbf{e}$ is the current enclosure of the error: these parts require only $\mathcal{O}(n^2)$ floating-point operations. The parts computed in higher precision thus take a negligible fraction of the whole computing time, which is dominated by the LU factorization, unless many iterations are required in order to reach the prescribed accuracy.

Newton method for univariate equations

If we now consider Newton method to solve a nonlinear univariate equation, we go even further: each iteration is a refinement of the current zero, and here we know that (except in the case of multiple roots, or otherwise ill-conditioned zeros) the convergence is quadratic. This means that at each step, the number of correct bits roughly doubles. It is thus easy to choose the computing precision: use a small one for the first iterations, until it limits the number of correct bits that can be reached, then increase the computing precision [5].

Conclusion

The following Julia libraries: `IntervalArithmetic.jl` [2], along with `IntervalLinearAlgebra.jl` [3], make it relatively easy to experiment the algorithms mentioned above.

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A $(max, +)$ based interval contractor for swarm localization

Luc Jaulin¹ and Laurent Hardouin²

¹ ENSTA, Lab-STICC
2 rue François Verny, 29200, Brest
luc.jaulin@ensta.fr

² Polytech Angers, LARIS
62 avenue notre-Dame du Lac, 49000 Angers
laurent.hardouin@univ-angers.fr

Keywords: Max plus algebra, Localization, Interval Methods

In this work, we propose an interval based approach [4] to improve the accuracy of the global localization and drastically reduces the uncertainties of the mutual positioning using the principle of DGPS (Differential Global Positioning Systems). Uncertainties will be represented by intervals.

Consider m robots $\mathcal{R}_i, i \in \{1, 2, \dots, m\}$, described by the following state equations:

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \\ \dot{v}_i \end{pmatrix} = \begin{pmatrix} v_i \cos \theta_i \\ v_i \sin \theta_i \\ \omega_i \\ \alpha_i \end{pmatrix} \quad (1)$$

where (x_i, y_i) is the position of $\mathcal{R}_i$, θ_i is its heading and v_i is the speed. The inputs are the yaw ω_i and the acceleration α_i .

We assume that for each time $t_k = k \cdot dt_k$, $k \in \{1, 2, 3, \dots\}$,

- each robot $\mathcal{R}_i$ measures its position $(x_i(k), y_i(k))$ with a precision $\pm e_{GPS}$ for each component. This means that for each t_k , we have $2m$ intervals $[x_i]$ and $[y_i]$ such that $x_i \in [x_i]$ and $y_i \in [y_i]$.
- for any pair of robots (i, j) , the vector

$$\begin{pmatrix} a_{ij}(k) \\ b_{ij}(k) \end{pmatrix} = \begin{pmatrix} x_j(k) - x_i(k) \\ y_j(k) - y_i(k) \end{pmatrix} \quad (2)$$

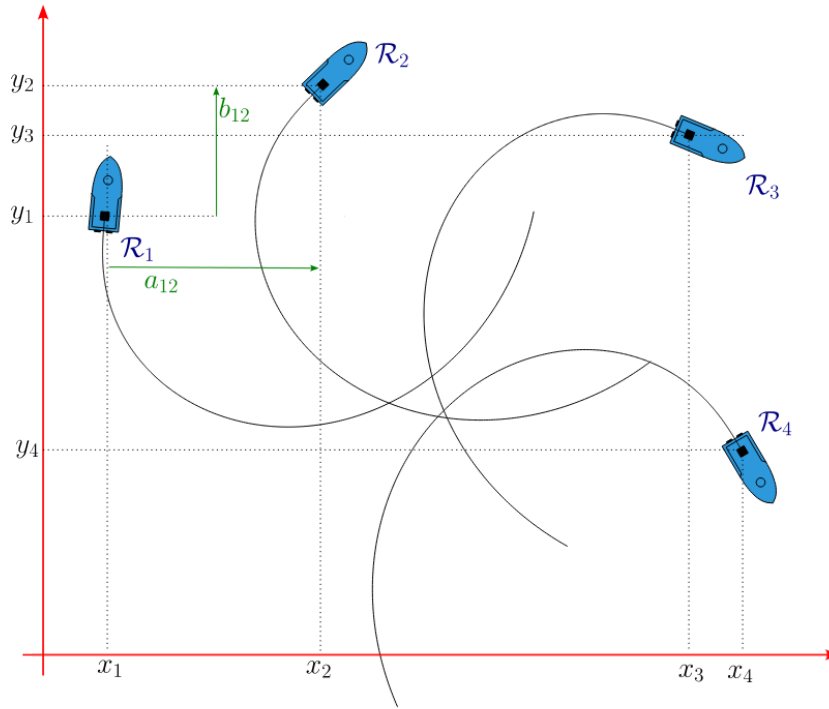
is measured with an error of $\pm e_{DGPS}$ for each component a_{ij}, b_{ij} . This means that for every t_k , we have $2m^2$ intervals $[a_{ij}]$ and $[b_{ij}]$ such that $a_{ij} \in [a_{ij}]$ and $b_{ij} \in [b_{ij}]$.

In our GNSS context, the size of the intervals $[a_{ij}], [b_{ij}]$ is small compared to the size of the intervals $[x_i]$ and $[y_i]$.

The figure below shows a situation with four mobile robots $\mathcal{R}_1, \dots, \mathcal{R}_4$. For times t_k , the m positions x_i, y_i and the m^2 mutual distances a_{ij}, b_{ij} are measured with some bounded uncertainties. Note that for each (i, j, ℓ) ,

$$\begin{aligned} a_{ij} &= a_{i\ell} + a_{\ell j} & (\text{Chasles}) \\ b_{ij} &= b_{i\ell} + b_{\ell j} \\ a_{ij} + a_{ji} &= 0 & (\text{symmetry}) \\ b_{ij} + b_{ji} &= 0 \end{aligned} \quad (3)$$

These relations can be used to contract the intervals for a_{ij}, b_{ij} . For this, we can take classical interval contractors [3], but the process would be slow due to the number of constraints that have to be considered (for Chasles: m^3 constraints for each t_k) and the propagation could be long. We would like to do better considering all these constraints as a global constraint and build a dedicated contractor to add to the catalog of global contractors [2]. In this work, we build this dedicated contractor using $(\max, +)$ algebra [1].



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Numerical Integration with Lie-Series and Result Verification

Christoph Spandl

Baden-Wuerttemberg Corporate State University Ravensburg
Campus Friedrichshafen, D-88045 Friedrichshafen, Germany
spandl@dhbw-ravensburg.de

Keywords: ODEs, Result Verification, Interval Methods

Introduction

In the present work, a method is presented for determining a verified upper bound on the global error of solutions of systems of ordinary differential equations. Here, the requirement is that the integration method for solving the system is given not only as a sequence of numbers referred to discrete times, but as a whole differentiable function, approximating the exact solution over an appropriate time interval. Integration methods under consideration include Taylor expansion based methods, like Lie series integration [1, 2] or neural networks based methods like physics informed neural networks [4].

Suppose some approximate solution $\tilde{x}(t)$ for $t \in I \subseteq \mathbb{R}$ on some open interval I is given, meeting the above requirements. The approximation solves the system of ordinary differential equations $\dot{x} = F(x)$ with $x \subseteq \mathbb{R}^n$ and some initial condition $x(t_0) = x_0$ where the global error of the approximate solution is unknown. Then in the following a method is presented, which computes afterwards a certified upper bound for the global error of the calculated solution. The global error is calculated either for the whole Interval I or at least for some open interval $I' \subseteq I$ containing $t_0 \in I'$.

Basic properties

The presented method is based on the calculation of the difference $\dot{\tilde{x}}(t) - F(\tilde{x}(t))$ which can, according to the requirements, be executed on the whole interval I . This difference represents some kind of local error and can be controlled by the integration method. It determines the accuracy of the approximation for each $t \in I$. In order to derive an upper bound for the global error, write $\tilde{x}(t) = \tilde{x}_0 + \int_{t_0}^t \dot{\tilde{x}}(\tau) d\tau$ with $\tilde{x}_0 = \tilde{x}(t_0)$, which can be written, since the approximation is demanded to be differentiable. It follows for the global error at time t :

$$\begin{aligned} |x_i(t) - \tilde{x}_i(t)| &\leq \int_{t_0}^t |\dot{x}_i(\tau) - \dot{\tilde{x}}(\tau)| d\tau + |x_{0_i} - \tilde{x}_{0_i}| \\ &\leq \max_{\tau \in [t_0, t]} \{|\dot{x}_i(\tau) - \dot{\tilde{x}}(\tau)|\} \cdot (t - t_0) + |x_{0_i} - \tilde{x}_{0_i}| \end{aligned}$$

for $i = 1, 2, \dots, n$ and $t \in I$. Finally, the equations of motion can be applied and gives

$$|\dot{x}_i(t) - \dot{\tilde{x}}(t)| \leq |F_i(x(t)) - F_i(\tilde{x}(t))| + |\dot{\tilde{x}}_i(t) - F_i(\tilde{x}(t))|$$

with the first term representing a recursion term and the second term controlled by the integration method.

Main results

To resolve the recursion, rewrite the above inequalities in the form $|\dot{x}_i(t + \Delta t) - \dot{\tilde{x}}(t + \Delta t)| - |\dot{x}_i(t) - \dot{\tilde{x}}(t)| \leq \Delta t \cdot \max_{\tau \in [t, t + \Delta t]} \{|F_i(x(\tau)) - F_i(\tilde{x}(\tau))|\} + \Delta t \cdot \max_{\tau \in [t, t + \Delta t]} \{|\dot{\tilde{x}}_i(\tau) - F_i(\tilde{x}(\tau))|\}$ by introducing a small time interval Δt with $[t, t + \Delta t] \subseteq I$. Applying Taylor's formula gives the right hand side $\Delta t \cdot \max_{\substack{\tau \in [t, t + \Delta t] \\ \Theta \in [0, 1]}} \left\{ \sum_k \left| \frac{\partial F_i}{\partial x_k}(x(\tau) + \Theta \cdot (\tilde{x}(\tau) - x(\tau))) \right| \cdot |x_k(\tau) - \tilde{x}_k(\tau)| \right\} + \Delta t \cdot \max_{\tau \in [t, t + \Delta t]} \{|\dot{\tilde{x}}_i(\tau) - F_i(\tilde{x}(\tau))|\}$. Introducing the global error function $\varepsilon_i(t) = |x_i(t) - \tilde{x}_i(t)|$ and the error function of the method $\varepsilon_{0_i}(t) = |\dot{\tilde{x}}_i(t) - F_i(\tilde{x}(t))|$, then executing the limit $\Delta t \rightarrow 0$ gives the differential inequality $\dot{\varepsilon}_i(t) \leq \sum_k \max_{\underline{\xi} \in B_{\underline{\varepsilon}(t)}(x(t))} \left\{ \left| \frac{\partial F_i}{\partial x_k}(\underline{\xi}) \right| \right\} \cdot$

$\varepsilon_k(t) + \varepsilon_{0_i}(t)$ for $i = 1, 2, \dots, n$. The tightest upper bound for the global error gives the *information equality*

$$\dot{\underline{\varepsilon}}(t) = M(\underline{x}(t)) \cdot \underline{\varepsilon}(t) + D(\underline{x}(t), \underline{\varepsilon}(t)) \cdot \underline{\varepsilon}(t) + \underline{\varepsilon}_0(t) \quad (1)$$

with the matrices $M(\underline{x}) = (m_{ij}(\underline{x}))$, $m_{ij} = \left| \frac{\partial F_i}{\partial x_j}(\underline{x}) \right|$ and $D(\underline{x}, \underline{\varepsilon}) = (d_{ij}(\underline{x}, \underline{\varepsilon}))$ with $d_{ij}(\underline{x}, \underline{\varepsilon}) = \max_{\xi \in B_{\underline{\varepsilon}}(\underline{x})} \left\{ \left| \frac{\partial F_i}{\partial x_j}(\xi) \right| - \left| \frac{\partial F_i}{\partial x_j}(\underline{x}) \right| \right\}$.

The right hand side of the information equality contains 3 terms. In the limit of vanishing errors, $\varepsilon \rightarrow 0$ and $\varepsilon_0 \rightarrow 0$, only the first term remains. Thus M represents the intrinsic dynamic instability of the underlying dynamis of the system. The generation of error due to the dynamic instability is related to the positive Liapunov exponents [5]. The second term is due to the actual uncertainty and represents error propagation. Even for a absolutely precise method, $\varepsilon_0 \rightarrow 0$, this term still remains. Finally the third term represents the local error created by the integration method.

The information equation is not only of theoretical interest. At first sight, obtaining a certified solution of a system of ODEs needs to solve the additional equation (1) which also needs to be certified. So it seems to be a circular approach. This obstacle can be overcome by calculating a poorer upper bound the following way. First, by interchanging $\underline{x}(t)$ and $\tilde{x}(t)$ when applying Taylor's formula, instead of (1), $\dot{\underline{\varepsilon}}(t) = M(\tilde{x}(t)) \cdot \underline{\varepsilon}(t) + D(\tilde{x}(t), \underline{\varepsilon}(t)) \cdot \underline{\varepsilon}(t) + \underline{\varepsilon}_0(t)$ can be derived. Second, set an a priori upper bound $\underline{\varepsilon}_{max}$ for the global error. This error is used in the second term of the above equation leading to $\dot{\underline{\varepsilon}}(t) = \tilde{M}(\tilde{x}(t)) \cdot \underline{\varepsilon}(t) + \underline{\varepsilon}_0(t)$ with $\tilde{M}(\tilde{x}(t)) = M(\tilde{x}(t)) + D(\tilde{x}(t), \underline{\varepsilon}_{max})$. This equation can be integrated analytically and the solution is $\underline{\varepsilon}(t) = \exp(\int_{t_0}^t \tilde{M}(\tilde{x}(\tau)) d\tau) \cdot \left(\int_{t_0}^t \exp(-\int_{t_0}^{\tau'} \tilde{M}(\tilde{x}(\tau)) d\tau) \cdot \underline{\varepsilon}_0(\tau') d\tau' + \underline{\varepsilon}(t_0) \right)$ which can be solved in a certified way by numerical integration with an interval method [6]. Suppose that the integration method is a Lie series expansion [2]. Then the local error $\underline{\varepsilon}_0(t)$ can be controlled by the order of the Lie series. A certified upper bound for the global error $\underline{\varepsilon}(t)$ at any time $t \in I'$ or a global upper bound on the whole interval I' can be calculated by subdividing the interval I' and calculating an interval enclosure of the

above integral ([6], Section 9.1). By appropriately choosing the subdivision, the enclosure can be sharpened. Finally, with this method it is also possible to determine a time interval $I'' \subseteq I'$, $t_0 \in I''$, with the property that the global error is bound by some a priori given value ε_{max} .

Some well known systems of ODEs are considered and numerical experiments are done to illustrate and validate the method.

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Information fusion enhanced state estimation for a class of nonlinear dynamical systems with unmodeled nonlinearities

Marit Lahme and Andreas Rauh

Carl von Ossietzky Universität Oldenburg
Distributed Control in Interconnected Systems
26111 Oldenburg, Germany
{Marit.Lahme, Andreas.Rauh}@uni-oldenburg.de

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Introduction

State estimation of dynamical systems is a cornerstone of modern control applications. State of the art techniques except data-driven approaches, typically rely on accurate system models to ensure reliable performance. In particular, methods such as observer-based state estimation [4, 5] and predictor-corrector frameworks [3] inherently depend on a known model structure to bound the system's state evolution within guaranteed sets. However, in real-world applications, physical phenomena such as friction, wear, or hysteresis effects are inherently difficult to capture with precise analytical models. We present a state estimation approach for a class of nonlinear dynamical systems, where the structure of the nonlinearity is not required to be known a priori. In contrast to unknown input estimation, this method focuses on identifying state-dependent characteristics. The proposed method is based on information fusion and combines multiple interval observers that incorporate the nonlinearity in the state equations in different ways. Additionally, these interval observers are combined with contractors

[1, 6] to improve the estimation accuracy. The core of the method is the identification of the nonlinearity by intersecting estimated interval boxes [2]. The knowledge of this identified characteristic is used to improve the estimation accuracy of the interval observers. Based on this method, interval observers can be designed that remain effective even in the presence of unmodeled nonlinear dynamics, thereby enhancing the applicability and reliability of state estimation in practical control scenarios.

Main results

As an example, we investigate the state estimation of a mass-spring-damper system with a nonlinear spring characteristic as shown in Fig. 1. The dynamic behavior of the system can be represented by

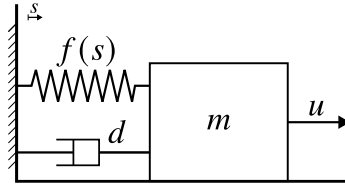


Figure 1: Mass-spring-damper system with a nonlinear spring characteristic $f(s) = -s - s^3$, where s is the spring elongation.

$$\dot{x}_1 = x_2 \quad , \quad (1)$$

$$\dot{x}_2 = f(x_1) - \frac{d}{m}x_2 + \frac{1}{m}u \quad , \quad (2)$$

with $x_1 = s$ and $x_2 = \dot{s}$. The structure of the spring characteristic $f(x_1)$ is not known beforehand. To incorporate this characteristic in the interval observers, it is modeled based on a quasi-linear representation, e.g. $f(x_1) = p(x_1)x_1$, $p(x_1) \in [P]$, where $[P]$ is an interval extension of a state-dependent model coefficient with known bounds. Fig. 2 presents a comparison between the estimation results obtained using the proposed method and those derived from a baseline approach

that relies solely on the quasi-linear representation without information fusion. The estimation accuracy can be enhanced significantly by applying the proposed method.

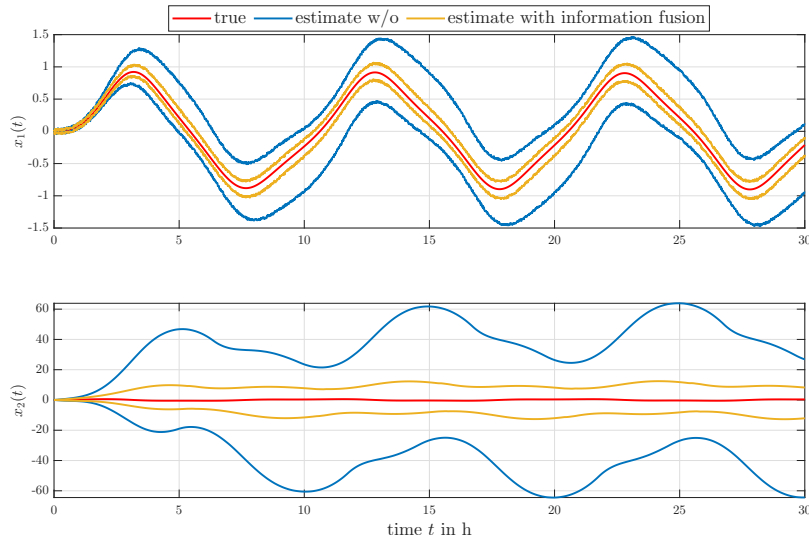


Figure 2: State estimation for the mass-spring-damper system with and without the proposed method.

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Characterizing indistinguishability sets in case of outliers

Clara Gondot^{1,2}, Fabrice Le Bars² and Luc Jaulin²

¹ ENSTA, Lab-STICC

² Rue François Verny, 29200 Brest

{clara.gondot,fabrice.le-bars,luc.jaulin}@ensta.fr

² CLEO-CMA

396 Avenue de la Roquette, 30200 Bagnols-sur-Cèze

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Indistinguishability set

In the context of set-membership estimation [5], we have a measurement vector $\mathbf{y}$ and we know that this vector is related to some unknown vector of interest $\mathbf{x}$ by the relation

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) + \mathbf{e} \quad (1)$$

where $\mathbf{e} \in \mathbb{E}$ is an unknown but bounded vector and $\mathbf{f}$ is a known function. The set of all feasible solutions is

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}) \quad (2)$$

where $\mathbb{Y} = \mathbf{y} - \mathbb{E}$.

We define the indistinguishability set $\hat{\mathbb{X}}(\mathbf{x})$ associated to $\mathbf{x}$ as the set of all $\hat{\mathbf{x}}$ that could produce the same measurement vector as $\mathbf{x}$. We have

$$\begin{aligned} \hat{\mathbb{X}}(\mathbf{x}) &= \{\hat{\mathbf{x}} \mid \exists \mathbf{e} \in \mathbb{E}, \exists \hat{\mathbf{e}} \in \mathbb{E}, \mathbf{f}(\mathbf{x}) + \mathbf{e} = \mathbf{f}(\hat{\mathbf{x}}) + \hat{\mathbf{e}}\} \\ &= \{\hat{\mathbf{x}} \mid \exists \mathbf{e} \in \mathbb{E}, \exists \hat{\mathbf{e}} \in \mathbb{E}, \mathbf{f}(\hat{\mathbf{x}}) = \mathbf{f}(\mathbf{x}) + \mathbf{e} - \hat{\mathbf{e}}\} \\ &= \{\hat{\mathbf{x}} \mid \mathbf{f}(\hat{\mathbf{x}}) \in \mathbf{f}(\mathbf{x}) + \mathbb{E} \ominus \mathbb{E}\} \\ &= \mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}) + \mathbb{E} \ominus \mathbb{E}) \end{aligned} \quad (3)$$

where $\ominus$ is the Minkowski difference. As a consequence, computing $\hat{\mathbb{X}}(\mathbf{x})$ corresponds to a set inversion problem. An algorithm such as SIVIA [4] can be used to compute an inner and an outer approximation of $\hat{\mathbb{X}}(\mathbf{x})$.

We would like to make this approach more robust by considering the case where outliers exist.

In case of outliers

In case of outliers [3], we generally assume that the set $\mathbb{E} = [e_1] \times \cdots \times [e_m]$ is a centered box and we reformulate the set inversion problem (2) as:

$$\mathbb{X} = \bigcap_{i=1}^{\{q\}} f_i^{-1}([y_i]) \quad (4)$$

where $[y_i] = y_i - [e_i]$ and q is the Maximum Number of Outliers (MNO). Using the transformation provided by (3) to get a set inversion definition for the indistinguishability set $\hat{\mathbb{X}}(\mathbf{x})$ is not possible yet. We need first to express (4) as the inverse of a set.

For this, let us introduce the pseudo-norm $\mathcal{L}_0$ used as a sparsity measure:

$$\mathcal{L}_0 : \mathbf{s} \mapsto \|\mathbf{s}\|_0 = \text{card}(\{i | s_i \neq 0\}) \quad (5)$$

Proposition 1. *Define the sparse set*

$$\mathbb{S}_q = \{\mathbf{s} \in \mathbb{R}^n \mid \|\mathbf{s}\|_0 \leq q\}$$

We have:

$$\bigcap_{i=1}^{\{q\}} f_i^{-1}([y_i]) = \mathbf{f}^{-1}(\mathbf{y} + \mathbb{E} \oplus \mathbb{S}_q)$$

Proof. The MNO assumption induces: $\exists \mathbf{s} \in \mathbb{S}_q, \mathbf{y} = \mathbf{f}(\mathbf{x}) + \mathbf{e} + \mathbf{s}$ $\square$

Proposition 2. *The indistinguishability set $\hat{\mathbb{X}}(\mathbf{x})$ can be expressed as the set inversion problem*

$$\hat{\mathbb{X}}(\mathbf{x}) = \mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}) + 2\mathbb{E} \oplus \mathbb{S}_{2q})$$

Proof. The proof uses the triangular inequality of the pseudo-norm $\mathcal{L}_0$ which yields the equality $\mathbb{S}_q \oplus \mathbb{S}_q = \mathbb{S}_{2q}$. $\square$

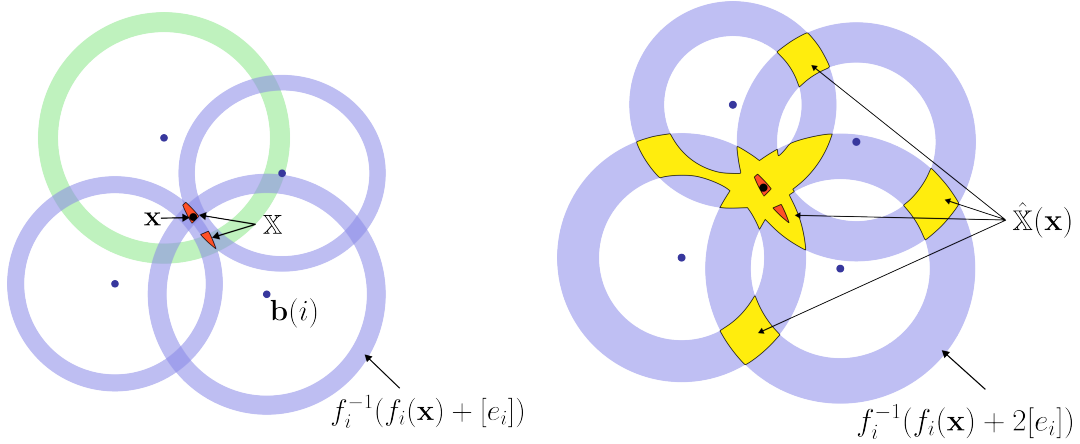
Corollary 3. *We have*

$$\hat{\mathbb{X}}(\mathbf{x}) = \bigcap_i^{\{2q\}} f_i^{-1}(f_i(\mathbf{x}) + 2\mathbb{E})$$

This formula allows us to use an efficient contractor approach [2][1] with the relaxed intersection operators for contractors.

Application

We apply the previous results to the 2-D localization of a point using distance measurements to multiple beacons, as illustrated below with 4 beacons and a MNO $q = 1$.



Left: 4 distance measurements generated by $\mathbf{x}$, the one in green (upper-left) being an outlier. In red the set inversion solution $\mathbb{X}$.

Right: The indistinguishability set $\hat{\mathbb{X}}(\mathbf{x})$ computed from the 4 true distances from $\mathbf{x}$ to the beacons $\mathbf{b}(i)$, taking into account the doubled bounded error and the MNO $q = 1$.

The position vector is

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T \quad (6)$$

And the observation functions are

$$f_i : \mathbf{x} \mapsto \sqrt{(x_1 - b_1(i))^2 + (x_2 - b_2(i))^2} \quad (7)$$

with $\mathbf{b}(i) = (b_1(i), b_2(i))$ the known positions of the i -th beacon.

Using contractors for the n distance constraints and the relaxed intersection, $\hat{\mathbb{X}}(\mathbf{x})$ is computed with different count of outliers and beacons in a simple interactive simulation.

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MAŁGORZATA MOCZURAD

Jagiellonian University

Rigorous Computation of Central Configurations via Interval Methods

Abstract

We will discuss our approach to the question of finding all central configurations. Our program performs an extensive top-down search for central configurations. At each step we bisect the box until a central configuration is found or proven not to exist. The algorithm succeeds until it stumbles upon bifurcations or masses close to zero. These cases have to be treated differently. At present we can deal with codimension one bifurcations and relatively thin intervals in the mass space.

With our program we obtained computer assisted proofs for some particular choices of masses for $n = 4, 5, 6, 7$ in planar case and $n = 5, 6$ in spatial case.

Joint work with Piotr Zgliczyński.

Trajectory-based search for periodic orbits in a memristive circuit and verification of their existence using interval methods

Łukasz Derwisz¹

¹ AGH University of Krakow
Faculty of Applied Mathematics
Krakow, Poland
derwisz@agh.edu.pl

Keywords: interval methods, rigorous numerics, periodic orbits, Poincaré map, memristive circuit, chaotic dynamics

Introduction

Periodic orbits are important objects in the analysis of chaotic dynamical systems, since they describe recurrent structures embedded in a chaotic attractor. In this work we study periodic orbits of the Poincaré return map associated with a nonlinear memristive circuit exhibiting chaotic dynamics. The circuit model is based on the double locally active memristor-based inductor-free chaotic circuit proposed in [1].

The aim of the work is to combine a trajectory-based numerical search with interval methods. The numerical part is used to locate candidates for periodic orbits, while the rigorous part is used to verify the existence of selected orbits by means of an interval Newton method. The general strategy follows the idea that a long trajectory of a chaotic system can be used to detect approximate recurrent behaviour and to generate good initial guesses for Newton-type refinement [2].

Trajectory-based search for periodic orbits

The continuous-time dynamics is reduced to a discrete-time problem by introducing a Poincaré section

$$\Sigma = \{(x_1, v_C, x_2) \in \mathbb{R}^3 : x_1 = x_1^*\}.$$

The corresponding Poincaré return map is denoted by P . In the numerical experiments a long trajectory of the return map is generated and the consecutive return points

$$z_0, z_1, z_2, \dots$$

on the section are collected.

For a prescribed period p , candidate periodic points are detected by searching for close returns along this trajectory. More precisely, points satisfying

$$\|z_{k+p} - z_k\| \leq \delta$$

for a chosen tolerance $\delta > 0$ are treated as approximate period- p returns. Such close returns provide initial guesses for Newton's method applied to the equation

$$F_p(z) = P^p(z) - z = 0.$$

A zero of F_p corresponds to a fixed point of P^p , and therefore to a periodic orbit of the return map.

After Newton refinement, the obtained candidates are filtered. First, duplicate candidates are removed, since the same orbit may be found several times from different close returns along the same long trajectory. Second, candidates corresponding to shorter periods are eliminated. This gives a list of numerically refined periodic orbits with prescribed minimal periods.

Interval verification

The second stage of the procedure is devoted to rigorous verification. For each Newton-refined candidate $\bar{z}$, a small interval box $[z]$ centered

at $\bar{z}$ is constructed on the Poincaré section. Then an interval Newton operator is applied to the equation

$$F_p(z) = P^p(z) - z = 0.$$

In the computations this has the form

$$N([z], \bar{z}) = \bar{z} - [DF_p([z])]^{-1}F_p(\bar{z}),$$

where $[DF_p([z])]$ denotes an interval enclosure of the derivative of F_p over the box $[z]$. If

$$N([z], \bar{z}) \subset [z],$$

then the existence of a true zero of F_p in $[z]$ is confirmed. Consequently, the existence of a periodic orbit of the Poincaré return map, and hence of the original continuous-time system, is rigorously verified. This type of interval Newton verification is closely related to rigorous computational techniques used for periodic orbits and periodic windows in chaotic systems [3].

In addition, the derivative of the full return map along the verified cycle is enclosed using interval arithmetic. This makes it possible to study the stability of the verified orbit. In particular, instability can be confirmed when the interval enclosure of a multiplier of P^p is separated from the unit circle.

Expected results

The main output of the computation is a structured list of periodic orbits for periods

$$p = 1, 2, \dots, 20.$$

For each period, the results will be summarized in a table containing the number of Newton-refined periodic orbits, the number of orbits verified by the interval Newton method, and the number of cases for which the verification was not completed with the current settings.

The obtained periodic-orbit counts may also be used to estimate the growth rate of periodic objects. If P_p denotes the number of periodic

orbits with minimal period p , then the number of fixed points of P^n can be reconstructed from the numbers of orbits with periods dividing n . These values can be used to compute finite-period estimates of the topological entropy of the Poincaré return map, following the approach based on periodic-orbit growth [2].

The presented approach therefore combines trajectory monitoring, Newton refinement and interval verification in order to obtain reliable information about periodic structures in a chaotic memristive circuit.

Acknowledgement

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Passive and Static Localization of an Electromagnetic Source using Interval Methods

Guillaume Garde, Lionel Lapierre and Luc Jaulin

ENSTA - Institut Polytechnique de Paris - Lab-STICC UMR 6285
2 rue François Verny, Brest, France
{guillaume.garde,lionel.lapierre,luc.jaulin}@ensta.fr

Keywords: Passive Localization, Swarm, Interval Methods

Introduction

Source localization is a key concept in various domains that exploit signal analysis techniques, such as telecommunication, astrophysics, robotics, or electronic warfare (EW) [4]. The latter aims to master the electromagnetic (EM) spectrum in defense contexts in order to maximize detection capabilities while remaining undetected [6]. Hence the need for efficient passive localization methods—that is, using incoming electromagnetic signals, such as RADAR signals, to localize the source without emitting anything to ensure stealth. Passive localization methods of EW have become quite accurate, but the corresponding equations are solved using estimators [9]. This means there is an associated uncertainty and a risk of losing valid solutions. A new approach using the same equations with interval methods is proposed here. It shows that it is possible to localize a source with guaranteed bounds using well-known passive localization principles.

Properties and Method

A peculiar attention is given to three different techniques. The Time Difference Of Arrival (TDOA) [7] [9], the Direction Of Arrival (DOA)

[5], and, for the specific context of scan-based emitters, the Time Difference Of Arrival Between Lobes (TDOA-BL) [1]. These methods can be combined to gain performance and accuracy depending on the scenario. Some hypotheses are made for this work. The emitter and the observers are considered to be static and to live in a two dimension plan. The problem of synchronization between observers is not addressed here. Future work will enable the use of more general assumptions.

Notation

Consider a two dimension cartesian plan. Let $P(x, y)$ be the emitter of unknown position. A network of n observers is represented by points $O_i(x_i, y_i), 0 \leq i \leq n$. Let c denote the speed of light. For an observer O_i , the time of arrival of a signal is t_i and the angle of arrival with respect to the North is α_i . In case of a scan-based rotating emitter, the rotation speed of the antenna is called w and is supposed to be constant.

TDOA

For a pair of observers (O_i, O_j) receiving the same EM signal the time difference of arrival equation is

$$t_i - t_j = \frac{1}{c}(\sqrt{(x - x_i)^2 + (y - y_i)^2} - \sqrt{(x - x_j)^2 + (y - y_j)^2}).$$

The solution of this equation is an hyperbola [2][9].

DOA

For an observer O_i receiving an EM signal, the direction of arrival equation is

$$\theta_i = \text{atan2}(y - y_i, x - x_i).$$

This equation gives a line passing through the source and the observer [5].

TDOA-BL

For a pair of observers (O_i, O_j) crossed through by a rotating EM beam, the time difference of arrival between lobes equation is

$$\alpha_i - \alpha_j = \text{atan2}(y - y_i, x - x_i) - \text{atan2}(y - y_j, x - x_j).$$

And

$$t_i - t_j = \frac{\alpha_i - \alpha_j}{\omega}$$

The solution is a circle arc [1].

Solving

In this work, these equations are solved using the SIVIA algorithm [3] for set inversion, and specific contractors and separators have been implemented using the CODAC library [8]. As the number of observers is increased, the solutions are given by the intersection of multiple sets. Different scenarios have been considered and played via a simulator. As sensors are not ideal, each measure or position is modeled by an interval whose length depends on the chosen uncertainty.

Main results

We provide here some computational results. Observers are represented by black boxes and the EM source by a red one. These boxes have been voluntarily enlarged for the purpose of representation. The solution sets are approached by an outside part (in blue), an inside part (in green), and bound (in orange).

Figures 1, 2 and 3 show the computed results of respectively a 3 observers TDOA scenario, a 3 observers DOA scenario and a 3 observers TDOA-BL scenario.

These methods are highly geometrical. Interval methods are particularly well suited for this. The major interest of this work is the

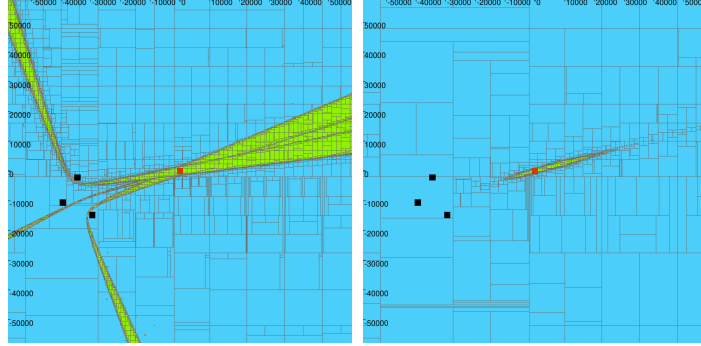


Figure 1: TDOA scenario

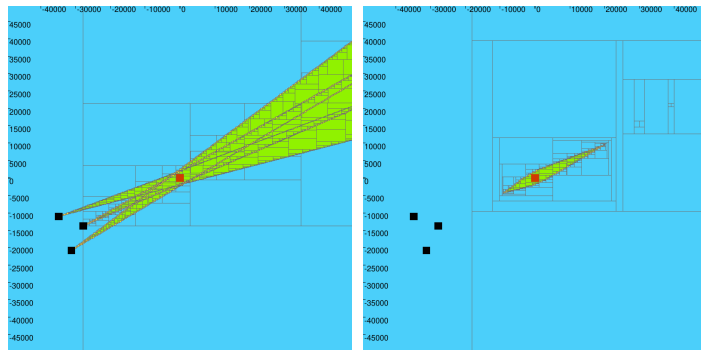


Figure 2: DOA scenario

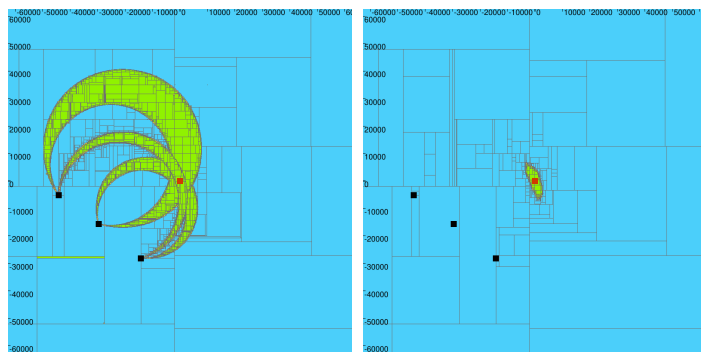


Figure 3: TDOA-BL scenario

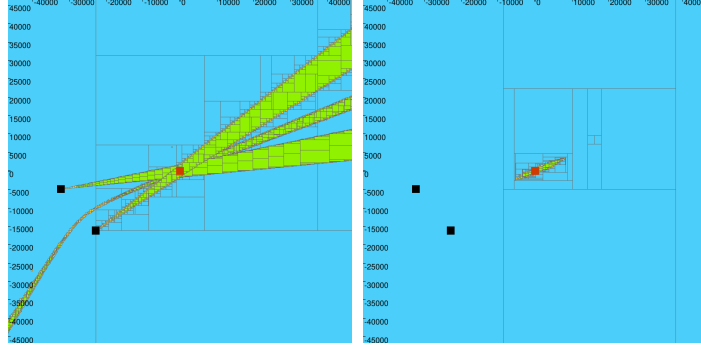


Figure 4: TDOA and DOA mixed scenario

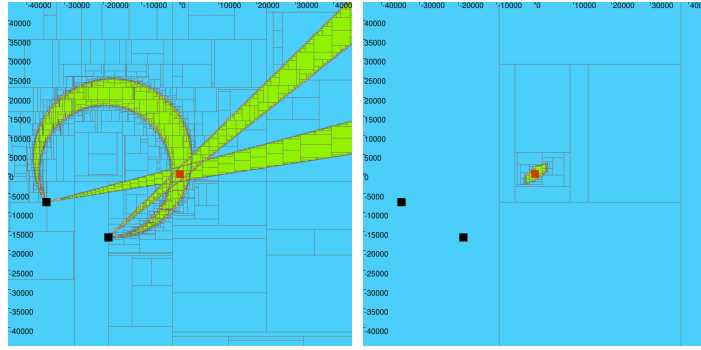


Figure 5: TDOA-BL and DOA mixed scenario

possibility of mixing them. Figures 4 and 5 show the computed results of 2 scenarios mixing first TDOA and DOA, then TDOA-BL and DOA. Only 2 observers are needed to obtain these results.

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SHIELD: Interval Bound Propagation for Certifiably Robust Continual Learning

Patryk Krukowski^{1,2,3}, Łukasz Gorczyca^{1,2}, Piotr Helm^{1,2},
Kamil Książek^{1,4} and Przemysław Spurek^{1,5}

¹ Jagiellonian University, Faculty of Mathematics and Computer Science
patryk.krukowski@doctoral.uj.edu.pl

² Jagiellonian University, Doctoral School of Exact and Natural Sciences
{lukasz.gorczyca, piotr.helm, kamil.ksiazek,

przemyslaw.spurek}@uj.edu.pl ³ Akces NCBR ⁴ Warsaw University of
Technology ⁵ IDEAS Research Institute

Keywords: Interval Bound Propagation, Interval Arithmetic, Certified Robustness, Neural Networks, Continual Learning, Wrapping Effect

Introduction

Interval Bound Propagation (IBP) [2] applies interval arithmetic [1] layer by layer through a neural network, producing rigorous output enclosures that certify robustness against adversarial perturbations within a given budget.

We present SHIELD [3], a framework that integrates IBP into a hypernetwork-based continual learning architecture, enabling certified adversarial robustness across a sequence of tasks without storing past data or gradients. As a core contribution, we introduce *Interval MixUp*, a training strategy that mitigates the wrapping effect [4] by controlling the radii of virtual interval-bounded inputs.

Interval propagation in neural networks

Consider a feedforward network of K layers, each applying $z_k = W_k z_{k-1} + b_k$, where W_k and b_k are the weights and biases in the $k \in \{1, \dots, K\}$

followed by a monotonic activation g . Given input $x \in \mathbb{R}^d$ ($z_0 = 0$) and perturbation radius $\varepsilon \geq 0$, the input enclosure is

$$I_\varepsilon(x) = \prod_{i=1}^d [x_i - \varepsilon, x_i + \varepsilon].$$

Using the midpoint μ_{k-1} and radius r_{k-1} , propagation gives:

$$\mu_k = W_k \mu_{k-1} + b_k, \quad r_k = |W_k| r_{k-1}, \quad (1)$$

where $|W_k|$ is the element-wise absolute value. For monotonic g : $\underline{z}_k = g(\mu_k - r_k)$, $\bar{z}_k = g(\mu_k + r_k)$. A prediction is *certifiably robust* if $\underline{z}_{K, y_{\text{true}}} > \max_{j \neq y_{\text{true}}} \bar{z}_{K, j}$.

SHIELD architecture

SHIELD consists of a **hypernetwork** $\mathcal{H}$ mapping trainable task embeddings $\mathbf{e}_t \in \mathbb{R}^N$ to weights $\boldsymbol{\theta}_t = \mathcal{H}(\mathbf{e}_t; \Phi)$ of a **target network** that processes interval-bounded inputs via IBP. Each task gets a dedicated classifier generated on demand, without storing past data.

The training loss combines: (i) standard cross-entropy on clean inputs, (ii) worst-case cross-entropy over IBP bounds (blended via an annealed κ), and (iii) a regularisation term penalising changes in the hypernetwork’s output for previous task embeddings, preventing catastrophic forgetting.

Interval MixUp: mitigating the wrapping effect

The wrapping effect [4] causes axis-aligned enclosures to grow overly conservative through nonlinear maps, weakening IBP’s certified guarantees. MixUp [5] trains on convex combinations

$$\tilde{x} = \lambda x_a + (1 - \lambda)x_b,$$

smoothing decision boundaries, but wrapping each $\tilde{x}$ in a full ε -ball exacerbates this effect. *Interval MixUp* scales the radius by proximity

to real data:

$$\varepsilon' = |2\lambda - 1| \cdot \varepsilon. \quad (2)$$

Near real samples ($\lambda \approx 0$ or 1): full radius; at the midpoint ($\lambda = 0.5$): $\varepsilon' = 0$. Smaller initial radii yield tighter enclosures after propagation, directly reducing the wrapping effect and closing the gap between classical and *verified* accuracy.

Robustness preservation across tasks

We prove that certified robustness is preserved across tasks: if the maximum logit change induced by a hypernetwork update $\mathbf{h}$ satisfies

$$\Delta_{\max}^{(s,t)}(x; \mathbf{h}) \leq \frac{1}{2}M(x, y_{\text{true}}; \boldsymbol{\theta}_{s,t}),$$

where M is the certified margin, then robustness guarantees for all previous tasks are maintained. The regularisation loss directly bounds this change.

Experimental results

We evaluate SHIELD on Permuted MNIST, Rotated MNIST, Split CIFAR-100, and Split miniImageNet (10 tasks each) under FGSM, PGD, and AutoAttack. SHIELD achieves state-of-the-art adversarial accuracy across all settings, reaching 57.9% under AutoAttack on Split miniImageNet versus 32.1% for the strongest baseline — nearly $2\times$ improvement — while maintaining competitive clean accuracy. Experiments on TinyImageNet (40 tasks) confirm scalability. Code: <https://github.com/pkrukowskii/SHIELD>.

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Towards Explainable and Fair Uncertainty Modeling in Healthcare via Set-Based Methods

Ekaterina Auer¹, Lorenz Gillner¹ and Wolfram Luther²

¹ University of Applied Sciences Wismar
Philipp-Müller-Straße 14, 23966 Wismar, Germany
{ekaterina.auer,lorenz.gillner}@hs-wismar.de

² University of Duisburg-Essen
Forsthausweg 2, 47057 Duisburg, Germany
wolfram.luther@uni-due.de

Keywords: Interval analysis, Dempster-Shafer theory, *BRCA1/2* mutation probability, Explainability, Fairness

Introduction

Explainability and fairness are widely recognized as key requirements for trustworthy healthcare prediction systems and AI applications more broadly. However, neither concept has a universally accepted definition [4, 8], and their practical implementation remains challenging. In [6], we propose a meta-level perspective on fairness (meta-fairness), in which explainability can be regarded as one of the quality criteria supporting fair decision-making. Alternatively, some conceptual frameworks, such as structured explanation taxonomies [10], include fairness as one of the components of explainability in AI systems.

In medical risk prediction, explainability and fairness are deeply intertwined with uncertainty. In particular, the way uncertainty is modeled and communicated not only affects the reliability of predictions but also their interpretability and the perceived fairness of decisions. While predictive models typically capture aleatory uncertainty associated with inherent variability, epistemic uncertainty arising from

incomplete knowledge of the data, model parameters, or model structure is often overlooked.

In this talk, we investigate how richer representations of epistemic uncertainty can improve the explainability of medical risk predictions and whether this increased transparency can support fairer decision-making. As a case study, we focus on estimating *BRCA1/2* mutation probabilities in hereditary breast and ovarian cancer syndrome. We compare a framework based on Dempster–Shafer theory [9] (DST) with logistic regression and a DST-based interpretation of logistic regression using a synthetic dataset. We extend this DST model using interval arithmetic [7], Yager functions [2], and thick intervals [1], emphasizing explicit epistemic uncertainty representation to enhance algorithmic transparency and support fairness at a meta-level. A light user study further explores how different uncertainty representations are perceived by non-expert users, providing insights into explainability.

Uncertainty versus Explainability and Fairness

Meaningful explanations should communicate not only the rationale behind a prediction but also its uncertainty. Without uncertainty information, predictions may create a false sense of precision, especially in settings with scarce or noisy data. Explicit epistemic uncertainty representations improve transparency by revealing model limitations and indicating when predictions should be interpreted with caution. Recent work has also argued that uncertainty itself can serve as an explainability mechanism [11].

The relationship between uncertainty and fairness is particularly important in healthcare, where predictive models support diagnosis, risk assessment, and treatment decisions. Due to differences in data availability, data quality, healthcare access, and historical exclusion, epistemic uncertainty is often unevenly distributed across patient populations. As a result, underrepresented groups may receive predictions based on weaker evidence than better-represented groups. Explicit uncertainty representation can help identify such limitations, revealing potential inequities in the underlying evidence base and supporting

fairer decision-making [3].

Although there is a growing body of work on the interplay between uncertainty and explainability or fairness [5], interval- and set-based approaches for epistemic uncertainty remain relatively underexplored in this context. Likewise, frameworks such as DST, which can represent both aleatory and epistemic uncertainty, have received limited attention from the perspective of their usefulness for explainability and fairness.

Summary

In this talk, we provide an overview of our work on representing epistemic uncertainty in hereditary breast and ovarian cancer syndrome risk prediction using interval analysis and DST. A particular challenge in this domain is that germline mutation status constitutes sensitive genetic information and is therefore typically unavailable in publicly accessible datasets. Consequently, we employ synthetic data, as our primary objective is to investigate uncertainty representation, algorithmic explainability, and fairness-related aspects rather than predictive performance itself. In particular, we examine how uncertainty modeling relates to explainability from an algorithmic perspective and to fairness through the lens of meta-fairness. Implementations of the proposed models, together with the synthetic datasets used in our experiments, are publicly available at

<https://github.com/lorenzgillner/BRCA-DST>.

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Verified Verification: Interval Methods in Proof Assistants

Pieter Collins¹

¹ Department of Advanced Computing Sciences
Maastricht University
Postbus 616
6200MD Maastricht
The Netherlands
`{pieter.collins@maastrichtuniversity.nl}`

Keywords: Proof Assistants, Constructive Analysis, Interval Arithmetic, Taylor Models, Dynamic Systems, Model Checking

Introduction

The aim of rigorous numerical methods such as those based on interval arithmetic is to give guaranteed bounds on the true value of the computed quantity. However, mistakes in the mathematical development of the method, or in the computational implementation, may mean that these guarantees are actually incorrect!

To obtain the best possible guarantees of correctness, we can develop our interval tools in a *proof assistant*, a software package which allows computational algorithms to be written and mathematical theorems about them stated and proved. The major proof assistants currently available are Rocq (formally Coq) [11], Lean [10] and Isabelle/HOL [9]. The proofs are checked by a small, trusted kernel. We call this process *verified verification* — a rigorous-numerics-based model-checking verification tool is itself verified by a logic-based proof assistant.

A motivating example for this work was a bug in the C++ source code of the ARIADNEtool [2] for reachability analysis and model-checking

of dynamic systems. This bug caused the error bounds of function approximations to be too low, and potentially invalid. This bug was not due to the algorithm itself, which had been formally verified [4], but in an incorrect efficiency optimization, illustrating the importance of verifying both code and implementation. As algorithms become more complicated, the potential to make mistakes is amplified.

Immler’s HOL-ODE-Numerics [7], which was used to verify [8] Tucker’s proof [12] of Smale’s 14th problem, is currently the largest body of formally-verified numerics for dynamical systems analysis.

Verified Rigorous Numerics

Our initial work on this project [4] focused on floating-point arithmetic and Taylor polynomial models in Rocq. We developed an abstraction of floating-point arithmetic using upward, downward and nearest rounded arithmetic, but without considering the machine epsilon, and used this to implement interval versions and the exponential function. We then implemented a data-type of Taylor polynomial models with floating-point coefficients, and proved correctness of arithmetical operations. The algorithms verified were those of ARIADNE, but the actual ARIADNE C++ code could not be verified.

The main results are that interval- and function-model based algorithms satisfy the *inclusion* property: If $\hat{X}$ is a computational data type, such as floating-point intervals, representing some mathematical type X , such as real numbers, then a function $f : X_1 \times X_2 \rightarrow Y$ is correctly modelled by a computational operation $\hat{f} : \hat{X}_1 \times \hat{X}_2 \rightarrow \hat{Y}$ if

$$\forall x_{1,2} \in X_{1,2}, \hat{x}_{1,2} \in \hat{X}_{1,2} \hat{x}_1 \ni x_1 \wedge \hat{x}_2 \ni x_2 \implies \hat{f}(\hat{x}_1, \hat{x}_2) \ni f(x_1, x_2).$$

In Rocq, this is described by:

Definition models {IX X : Type} : IX -> X -> Prop.

Proposition op_correct
 (iop : IX1 -> IX2 -> IY) (op : X1 -> X2 -> Y) :=
 forall (ix1 ix2 : IX) (i1 i2 : X),
 models mx1 x1 -> models mx2 x2 ->
 models (iop ix1 ix2) (op x1 x2).

In future work we will to incorporate genuine floating-point arithmetic such as with Rocq’s Flocq library [6]. Once this is in place, we will start developing full model-checking functionality, including differential equations and constraint satisfaction.

Verified Computable Analysis and Systems Theory

To support the rigorous numerics, we are also working on formalising the underlying theory of *computable analysis*, which shows what is possible from arbitrarily-accurate rigorous numerics. We are investigating an approach of finding logical axioms which make the constructive operations of Rocq coincide with the Turing-computable functions. Core data types include the compactified natural number type $\mathbb{N}_\infty$, the Cantor type $\{0, 1\}^\mathbb{N}$ and the Sierpinski type $\{\top, \bot\}$ as classifier for verifiable predicates. Results of [5] show that with the correct axioms, one can show that every constructible function $\mathbb{N} \rightarrow \{0, 1\}$ is continuous.

To support applications to systems theory, we are working on formalising compositional theory of input-output systems [3], with the ultimate goal of developing the theory for hybrid systems [1], which is sufficiently complicated that we are not sure all the results are correct. As part of this effort, we have work on timed-event systems and nondeterministic systems using *monads* as a model of nondeterminism. The formalisation effort uncovered a deep but subtle barrier to developing a compositional systems theory for stochastic systems.

Perspectives

We expect the use of proof assistants to become increasingly prevalent in mathematics and scientific computation. For safety-critical systems and use in complicated mathematical proofs, the benefits of having complete confidence in correctness is very appealing. Further, proof assistants synergise well with generative artificial intelligence, since they will catch machine-generated hallucinations and other mistakes, and encourage collaboration by giving a precisely-specified common frame of reference. The functional nature of these tools may mean that

efficiency is an issue, but this can probably be largely circumvented by designing efficient functional algorithms and exporting to compiled languages, such as Rocq to OCaml.

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A Comparison between the Theorem of Ehlich and Zeller and Polytopic Uncertainty Representations for the Development of Interval Observers

Andreas Rauh¹, Fabian Schneider², Marit Lahme¹, and
Christian Wölfel²

¹ Carl von Ossietzky Universität Oldenburg
Distributed Control in Interconnected Systems
D-26111 Oldenburg, Germany
{Andreas.Rauh, Marit.Lahme}@uni-oldenburg.de

² Ruhr University Bochum
Chair of Automation
D-44780 Bochum, Germany
{Fabian.Schneider, Christian.Woelfel}@atp.rub.de

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Interval Observer Design for Linear Systems

Interval observers provide a means to perform state estimation in a bounded error framework, where lower and upper bounding trajectories are determined so that the sets of reachable states are enclosed at each point in time despite bounded uncertainty in the dynamic system model as well as in the measurement process [2, 5].

Originally, this design was developed for linear time-invariant (LTI) system models

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \mathbf{w}(t) \quad (1)$$

$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{v}(t) \quad (2)$$

Here, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ the control vector, and $\mathbf{y}(t) \in \mathbb{R}^q$ the vector of measurements. In addition, $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are bounded disturbances in the form $\mathbf{w}(t) \in [\underline{\mathbf{w}}(t) ; \overline{\mathbf{w}}(t)]$ and $\mathbf{v}(t) \in [\underline{\mathbf{v}}(t) ; \overline{\mathbf{v}}(t)]$, respectively.

In the original publications on interval observer design for LTI systems (1) with the uncertain initial conditions $\mathbf{x}(0) \in [\underline{\hat{\mathbf{x}}}(t) ; \overline{\hat{\mathbf{x}}}(t)]$, the observer was defined as

$$\dot{\underline{\hat{\mathbf{x}}}}(t) = \mathbf{A} \cdot \underline{\hat{\mathbf{x}}}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \underline{\mathbf{w}}(t) + \mathbf{H} \cdot (\mathbf{y}_m(t) + \underline{\mathbf{v}}(t) - \mathbf{C} \cdot \underline{\hat{\mathbf{x}}}(t)) \quad (3)$$

$$\dot{\overline{\hat{\mathbf{x}}}}(t) = \mathbf{A} \cdot \overline{\hat{\mathbf{x}}}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \overline{\mathbf{w}}(t) + \mathbf{H} \cdot (\mathbf{y}_m(t) + \overline{\mathbf{v}}(t) - \mathbf{C} \cdot \overline{\hat{\mathbf{x}}}(t)) \quad (4)$$

with the sensor measurements $\mathbf{y}_m(t)$. For decoupling the lower and upper bounding trajectories of the estimator according to (3) and (4), it is typically assumed that the lower state bound $\underline{\hat{\mathbf{x}}}(t)$ and $\mathbf{B} \cdot \mathbf{u}(t)$ are element-wise non-negative and that all off-diagonal elements of $\mathbf{A} - \mathbf{H}\mathbf{C}$ are non-negative, i.e., this matrix is Metzler. Moreover, stability needs to be ensured by a suitable choice of $\mathbf{H}$.

In recent years, generalizations were published for systems with bounded uncertainty in the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ and for systems with negative bounds $\underline{\hat{\mathbf{x}}}(t)$. Moreover, relaxations of the Metzler property of $\mathbf{A} - \mathbf{H}\mathbf{C}$ become possible by either coordinate transformations [4] or by extending the observer differential equations (3) and (4) by the so-called TNL approach, cf. [7].

Generalization to Nonlinear System Models

For nonlinear system models, a polytopic (or norm-bounded) uncertainty model is typically determined to enclose the variability of all state-dependent dynamics after re-writing the system model in the quasi-linear representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t)) \cdot \mathbf{u}(t) + \mathbf{w}(t) \quad (5)$$

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{x}(t)) \cdot \mathbf{x}(t) + \mathbf{v}(t) \quad (6)$$

As discussed in [6], such kind of polytopic uncertainty representation of the state-dependent matrices in the quasi-linear model (5)

easily leads to enclosing many combinations of values that are not at all attainable during the evolution of the system.

As shown in [1], the theorem of Ehlich and Zeller [3] — that was originally developed to determined range enclosures for polynomial expressions over a finite range of their input argument(s) — can be exploited as an alternative.

Consider a univariate polynomial

$$p(x) = \sum_{i=0}^d a_i x^i \quad (7)$$

of degree d with $x \in [\mathcal{X}] = [\underline{x} ; \bar{x}]$ and determine the Chebychev points $x^{(i)} = \frac{\underline{x} + \bar{x}}{2} + \frac{\bar{x} - \underline{x}}{2} \cos\left(\frac{(2i-1)\pi}{2N}\right)$, where $N > d$, $N \in \mathbb{N}$. Then, the bounds

$$\begin{aligned} \sup_{x \in [\mathcal{X}]} p(x) &\leq \frac{1}{2} \left((C_p + 1) p_{\max}^{\mathcal{X}^{(N)}} - (C_p - 1) p_{\min}^{\mathcal{X}^{(N)}} \right) \quad \text{and} \\ \inf_{x \in [\mathcal{X}]} p(x) &\geq \frac{1}{2} \left((C_p + 1) p_{\min}^{\mathcal{X}^{(N)}} - (C_p - 1) p_{\max}^{\mathcal{X}^{(N)}} \right) \end{aligned} \quad (8)$$

with

$$\begin{aligned} p_{\min}^{\mathcal{X}^{(N)}} &= \min_{i \in \{1, \dots, N\}} p\left(x^{(i)}\right) \quad \text{and} \\ p_{\max}^{\mathcal{X}^{(N)}} &= \max_{i \in \{1, \dots, N\}} p\left(x^{(i)}\right), \end{aligned} \quad (9)$$

where

$$C_p = \left(\cos\left(\frac{d\pi}{2N}\right) \right)^{-1} \quad (10)$$

enclose the complete range of the function $p(x)$ over its input domain.

In the frame of an interval observer synthesis, the benefit of this approach is the ability to better capture correlations that exist between the elements of the state-dependent matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{C}(\mathbf{x})$. In this contribution, we investigate this procedure not only for univariate polynomials but also for the multi-variate case and thoroughly analyze the influence of this type of range computation over the classically employed polytopic approach.

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Global changes in the structure of maximal invariant sets

Daniel Wilczak¹, Sergio Serrano² and Roberto Barrio²

¹ Jagiellonian University

Faculty of Mathematics and Computer Science
ul. Prof. Łojasiewicza 6, 30-348 Kraków, Poland

`Daniel.Wilczak@uj.edu.pl`

² University of Zaragoza

Departamento de Matemática Aplicada and IUMA. E-50009. Spain.
`{sserrano,rbarrio}@unizar.es`

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Introduction

Attracting invariant sets are easily observable in numerical simulation. On the other hand, understanding internal dynamics on such invariant set is challenging. In particular, the observed maximal invariant may not be minimal. A typical situation is when an attracting cycle coexists with a chaotic set and a very long transition time is needed to observe the convergence to the periodic solution.

We present a computational framework for verification that a dissipative ODE admits a compact and connected maximal invariant set. We also provide a methodology to bound from below complexity of the dynamics – a lower bound on the topological entropy of the system is computed by means of interval-based methods. We also show, that local bifurcations of periodic orbits may lead to changes in global structure of the maximal invariant set enriching global orbit structure.

All main results are based on [4] and they are computer-assisted. We use the CAPD library [1] to compute bounds on the solutions to underlying ODEs, Poincaré maps [2] and their partial derivatives.

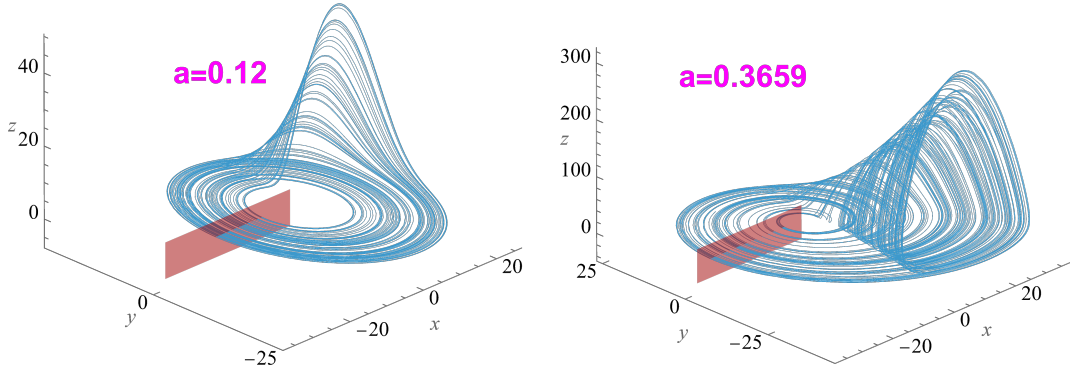


Figure 1: Poincaré section and a typical trajectory of (1) for $a = \min \mathcal{A}$ (left) and $a = \max \mathcal{A}$ (right).

Main results

Consider the Rössler system [3]

$$\dot{x} = -(y + z), \quad \dot{y} = x + ay, \quad \dot{z} = b + z(x - c) \quad (1)$$

with fixed $b = 0.2$ and $c = 15$. We analyse (1) for an explicit range of parameter values $a \in \mathcal{A} := [0.12, 0.3659]$. Denote by

$$\Pi := \{(x, y, z) \in \mathbb{R}^3 : y = 0 \wedge \dot{y} = x < 0\}, \quad P_a : \Pi \rightarrow \Pi$$

a Poincaré section and the corresponding Poincaré map, respectively. We will use (x, z) coordinates on Π . Numerical simulation shows that for a range of parameter values a large maximal invariant set of (1) exists and it changes its structure with the parameter – see Figure 1.

The following theorem summarizes results from [4] about the dynamics of (1).

Theorem 1. Put $\mathcal{T} := [-30.53, -3] \times [0.004, 0.011] \subset \Pi$.

1. For $a \in \mathcal{A}$, the Poincaré map P_a is well defined and smooth on $\mathcal{T}$ and $P_a(\mathcal{T}) \subset \text{int}\mathcal{T}$ – see Figure 2.
2. P_a undergoes a saddle-node bifurcation at the points (x_i^*, z_i^*, a_i^*) , $i = 1, \dots, 5$ with

$$|x_i^* - x_i| \leq 3 \cdot 10^{-11}, \quad |z_i^* - z_i| \leq 10^{-14}, \quad |a_i^* - a_i| \leq 6 \cdot 10^{-14},$$

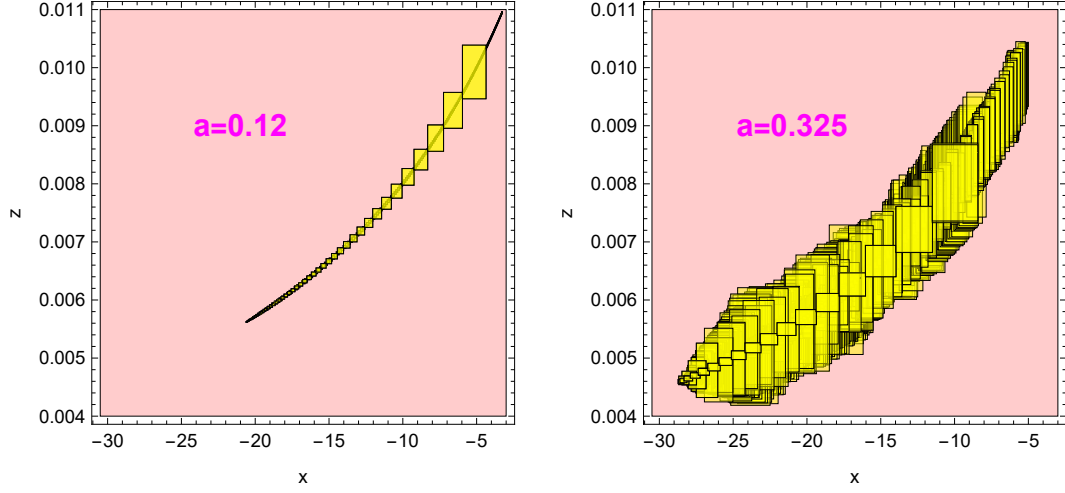


Figure 2: Trapping region $\mathcal{T}$ (in pink) and a bound on $P_a(\mathcal{T})$ (in yellow) for two selected parameter values.

where

i	x_i	z_i	a_i
1	-24.98615641824929	0.005003695953767296	0.2445890212249042
2	-27.90101006311546	0.004663547021688063	0.3119866509093180
3	-29.22211573599117	0.004524155280447693	0.3405236989996505
4	-29.89377365336397	0.004456435009829962	0.3546641965836549
5	-30.25817997684925	0.004420535075303823	0.3623180183723328

The branches of periodic points continue to exist at least to $a = \max \mathcal{A}$ – see Figure 3 (bottom).

3. For $a \in \mathcal{A}$ there is an invariant set $\mathcal{H}_a \subset \Pi$. The mapping $P_a|_{\mathcal{H}_a} : \mathcal{H}_a \rightarrow \mathcal{H}_a$ is semiconjugated to a subshift of finite type, with the number of symbols varying between 3 and 12. The topological entropy of $P_a|_{\mathcal{H}_a}$ is growing with the parameter reaching $h_{top}(P_{\max \mathcal{A}}) \geq \ln 12$ – see Figure 3 (top).

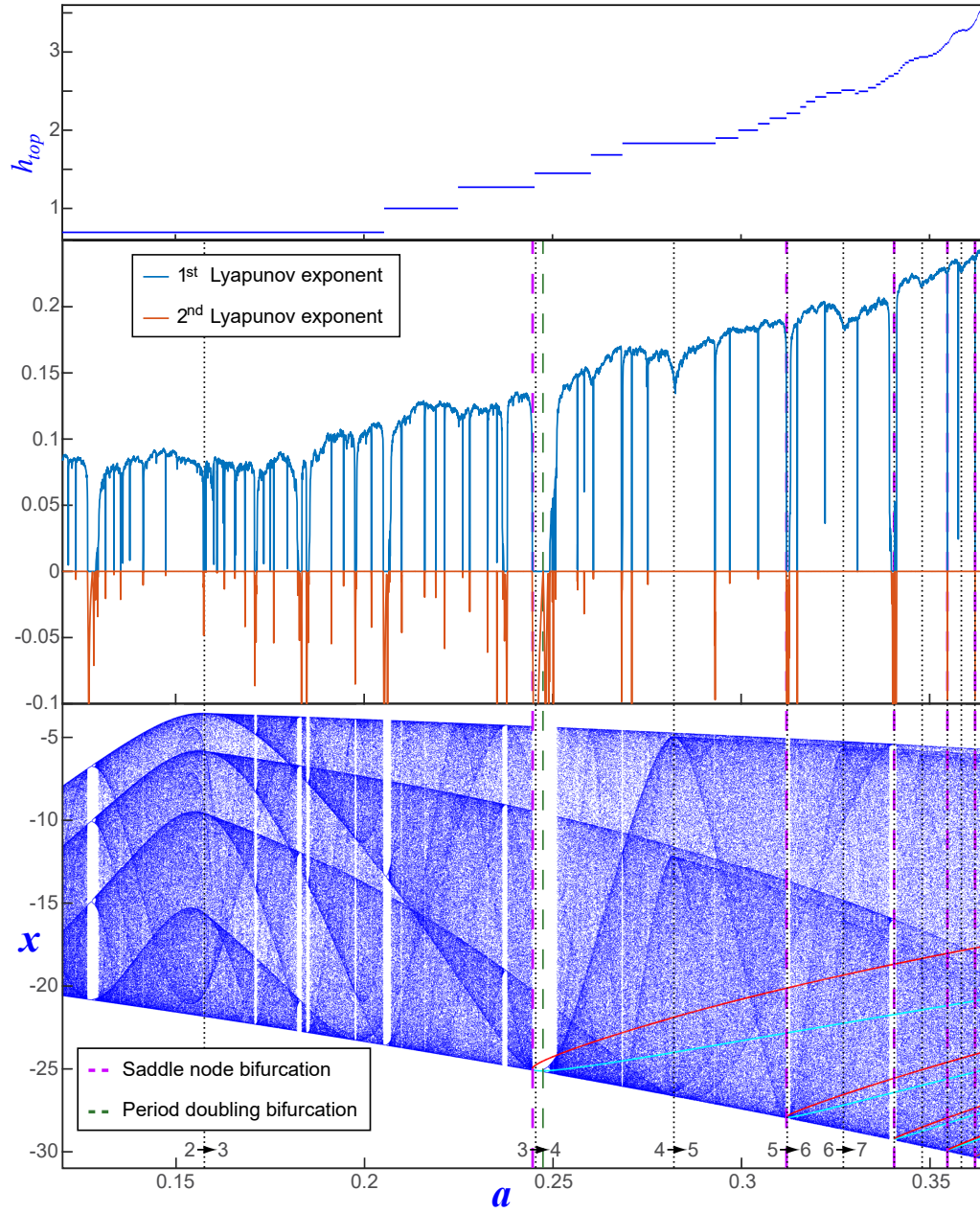


Figure 3: Top: lower bound on the topological entropy. Middle: Lyapunov exponents. Bottom: Bifurcation diagram representing the x -coordinate of the intersection points of the attractor with the Poincaré section Π . The dashed vertical segments indicate some saddle-node (in purple) and period-doubling (in green) bifurcations. Blue and red curves show stable and unstable families of periodic orbits born from the previous saddle-node bifurcations.

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Rigorous integration of dissipative PDEs with periodic boundary conditions

Piotr Zgliczyński

Jagiellonian University, Institute of Computer Science and Computational Mathematics
ul. Łojasiewicza 6, 30-348 Kraków, Poland.
`umzglicz@cyf-kr.edu.pl`

We explain how to do rigorous numerics for dissipative PDEs with periodic boundary conditions and how to use finite dimensional tools from dynamics in the context of dissipative PDEs. This will be illustrated on the example of the proof of the existence of symbolic dynamics for the Kuramoto-Sivashinsky PDE on the line: a semiconjugacy with the Bernoulli shift, the existence of countably many periodic orbits and homo- and heteroclinic connections between them. [1, 2, 3]

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Rigorous C^0 and C^1 integration of dissipative PDEs

Jakub Banaśkiewicz

Jagiellonian University
ul. Profesora Łojasiewicza 6
jakub.banaskiewicz@uj.edu.pl

Keywords: Rigorous Integration, Interval Methods, C^1 computations, Dissipative PDEs

Abstract

In this talk, I will introduce rigorous integration methods for dissipative PDEs and their application in computer-assisted proofs. These methods can provide rigorous bounds for the evolution of infinite-dimensional dynamical systems. First, I will present the C^0 rigorous integration algorithm, originally developed for the Kuramoto–Sivashinsky equation in [9, 10] and subsequently applied to the Burgers, Swift–Hohenberg and Brusselator equations. Building on this, I will discuss its extension to a C^1 algorithm, which additionally controls the derivative of the flow with respect to initial conditions. Such control is essential for validating hyperbolicity, local attraction and for applying Krawczyk method in infinite dimensions. I briefly discuss earlier C^1 approaches — the pioneering computation for a hyperbolic periodic orbit in the Kuramoto–Sivashinsky equation [1], the logarithmic-norm technique for the non-autonomous Burgers equation [11], and the recent block-splitting method [7, 8]. Then, I present my method in which we work directly with the variational equation. The C^0 algorithm is applied to a suitably constructed set $X^{h,0}$ for the variational part, whose elements may even be unbounded in the phase space. By linearity

we can later combine results for different directions, obtaining point-wise estimates of $D_{x_0}\varphi(t, t_0, X^0)h^0$ for all h^0 in the tail of the Fourier expansion.

Two benchmark problems will illustrate the method.

- For the **Burgers equation with a fractional Laplacian**, we integrate a set of the form $X^{h,0} = \{u \in H_{\text{odd}}^2(-\pi, \pi) : u_k \in k^{-2}[-1, 1], k \in \mathbb{N}\}$, which contains a ball in H_{odd}^2 . This directly yields the proof of a locally attracting periodic orbit.
- For the **non-autonomous Chafee–Infante equation**

$$\begin{cases} u_t = u_{xx} + \lambda u - b(t)u^3, & (x, t) \in (0, \pi) \times (t_0, \infty), \\ u(t, x) = 0, & (x, t) \in \{0, \pi\} \times (t_0, \infty), \\ u(t_0, x) = u^0(x), & x \in (0, \pi), \end{cases}$$

we integrate *unbounded* sets such as $X^{h,0} = \{u \in C_0(0, \pi) : u_k \in [-1, 1], k \in \mathbb{N}\}$ or even $u_k \in [-k^p, k^p]$ with $p > 0$. This provides effective control of the derivative over every Fourier mode and leads to the existence of a locally attracting periodic orbit.

The talk will conclude with a discussion of ongoing and future work: rigorous estimation of Poincaré map derivatives for autonomous systems, the use of the C^1 algorithm within Newton–Krawczyk methods to prove existence of unstable periodic orbits, and a deeper study of attractor structures in the Chafee–Infante and Brusselator models, including heteroclinic connections and period-doubling bifurcations. All computer-assisted proofs are implemented using the CAPD library [5], and the source codes are publicly available [4, 3].

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Parameterization Approaches for Luenberger-Type Contractors in Set-Based State Estimation

Andreas Rauh and Marit Lahme

Carl von Ossietzky Universität Oldenburg
Distributed Control in Interconnected Systems
D-26111 Oldenburg, Germany

`{Andreas.Rauh, Marit.Lahme}@uni-oldenburg.de`

Keywords: Interval observer design, Bounded error state estimation, Zonotopic state enclosures, Contractors

Approaches for Bounded Error State Estimation

Typically, set-based state estimation approaches for dynamic systems with bounded uncertainty are classified into predictor–corrector approaches [6] and interval observers [3]. Predictor–corrector approaches rely on forecasting reachable domains in the state-space using the system’s open-loop dynamics until the time instant at which measured data become available. Then, the predicted sets are contracted with the help of measurement information which is also assumed to be subject to bounded uncertainty. As shown in [1, 5], this contraction procedure can be replaced by a Luenberger-like observer approach. Computationally efficient implementations can be achieved if the sets of reachable states are represented with the help of zonotopes. In contrast, interval observers commonly assume the direct computation of lower and upper bounding trajectories for each of the state variables. Due to the required assumptions on monotonicity of the error dynamics, the design conditions are subject to stronger constraints than in the case of Luenberger-type contractors.

Restrictions of Interval Observer Synthesis

To relax the restrictions that hold due to monotonicity properties [3], approaches relying on coordinate transformations have been introduced. In general, these approaches transform a non-cooperative realization into a cooperative one [2]. Alternatively, extensions of the observer structure by further design degrees of freedom [7] have been developed which do not only account for measurements at a single point in time but which add two further gain matrices and a measurement at an additional time instant. Typically, these approaches are parameterized with the help of linear matrix inequality techniques, where optimality of the solution is achieved by a minimization of the worst-case amplification within the signal path from the uncertain external influence factors to the estimated state enclosures.

In this paper, we replace the design criteria for this so-called TNL observer technique (named after the three design matrices) by an optimization that is inspired by the minimization of the one time-step estimation error co-variance as it is done in the frame of the design of Kalman filters for linear dynamic systems. Generalizing this approach to the TNL observer according to the following strategy allows for more easily weighting the magnitudes in the individual uncertain influence factors in the process and measurement models. However, cooperativity is not necessarily obtained so that the implementation is then performed with the help of a zonotopic state contractor. To get an impression when this type of parameterization is advantageous in comparison to the classical TNL design wrt. existence of solutions and tightness of the computed state enclosures, different numerical case studies are presented on the basis of system models that are inspired by real-life applications.

Sketch of the Gain Optimization Procedure

Assume a discrete-time state estimator given in the form [7]

$$\hat{\mathbf{x}}_{k+1} = \mathbf{T}\mathbf{A}\hat{\mathbf{x}}_k + \mathbf{T}\mathbf{B}\mathbf{u}_k + \mathbf{L} \cdot (\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_k) + \mathbf{\Delta}_k + \mathbf{N}\mathbf{y}_{k+1} \quad (1)$$

with the state estimate $\hat{\mathbf{x}}_k$, the control signal $\mathbf{u}_k$, the measurement $\mathbf{y}_k$, and the disturbances Δ_k resulting from process and measurement noise. Assume further that $\mathbf{T} = \mathbf{I} - \mathbf{N}\mathbf{C}$ holds, where $\mathbf{I}$ is the identity matrix of appropriate dimensions, and $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{L}$ are the design degrees of freedom of the observer, cf. [7].

In contrast to the classical parameterization of this observer, we do not aim at minimizing an ∞ -norm criterion but rather optimize a quadratic norm on the estimation error dynamics given by

$$\mathbf{e}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k, \quad (2)$$

where $\mathbf{x}_k$ is the true, however, not directly measurable system state. Using the ansatz (1) and assuming that the dynamics matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are accurately known, the error dynamics are then given by

$$\begin{aligned} \mathbf{e}_{k+1} &= (\mathbf{T}\mathbf{A} - \mathbf{L}\mathbf{C}) \mathbf{e}_k + \Delta_k \\ &= (\mathbf{A} - \mathbf{N}\mathbf{C}\mathbf{A} - \mathbf{L}\mathbf{C}) \mathbf{e}_k + \Delta_k. \end{aligned} \quad (3)$$

If the uncertainties for state reconstruction and external perturbations are assumed to be normally distributed, an optimal observer parameterization is obtained by the minimization of the one-time-step error covariance

$$\begin{aligned} \mathbb{E} \{ \mathbf{e}_{k+1} \mathbf{e}_{k+1}^T \} &= (\mathbf{A} - \mathbf{N}\mathbf{C}\mathbf{A} - \mathbf{L}\mathbf{C}) \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot (\mathbf{A} - \mathbf{N}\mathbf{C}\mathbf{A} - \mathbf{L}\mathbf{C})^T \\ &\quad + (\mathbf{I} - \mathbf{N}\mathbf{C}) \mathbf{E} \cdot \mathbb{E} \{ \mathbf{w}_k \mathbf{w}_k^T \} \cdot \mathbf{E}^T (\mathbf{I} - \mathbf{N}\mathbf{C})^T \\ &\quad + \begin{bmatrix} \mathbf{L} & \mathbf{N} \end{bmatrix} \cdot \begin{bmatrix} \mathbb{E} \{ \mathbf{v}_k \mathbf{v}_k^T \} & \mathbf{0} \\ \mathbf{0} & \mathbb{E} \{ \mathbf{v}_{k+1} \mathbf{v}_{k+1}^T \} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{L}^T \\ \mathbf{N}^T \end{bmatrix}. \end{aligned} \quad (4)$$

Here, the term $\mathbf{E}\mathbf{w}_k$ represents the influence of the process uncertainty $\mathbf{w}_k$ on the state update $\mathbf{x}_{k+1}$; $\mathbb{E} \{ \mathbf{w}_k \mathbf{w}_k^T \}$ is the process noise covariance; $\mathbb{E} \{ \mathbf{v}_k \mathbf{v}_k^T \}$ and $\mathbb{E} \{ \mathbf{v}_{k+1} \mathbf{v}_{k+1}^T \}$ are the measurement noise covariances related to additive uncertainty of the measurements $\mathbf{y}_k$ and $\mathbf{y}_{k+1}$, respectively.

Now, an optimization of the gain matrices $\mathbf{L}$ and $\mathbf{N}$ becomes possible by a minimization of the trace

$$\text{tr} \left(\mathbb{E} \{ \mathbf{e}_{k+1} \mathbf{e}_{k+1}^T \} \right) \quad (5)$$

of the estimation error covariance. The necessary optimality conditions result from setting the respective partial derivatives with respect to $\mathbf{L}$ and $\mathbf{N}$ to zero, leading to the linear set of equations

$$\begin{aligned} & \left[\mathbf{A} \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot (\mathbf{CA})^T + \mathbf{E} \cdot \mathbb{E} \{ \mathbf{w}_k \mathbf{w}_k^T \} \cdot \mathbf{C}^T \mid \mathbf{A} \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot \mathbf{C}^T \right] \\ &= [\mathbf{N} \quad \mathbf{L}] \cdot \\ & \left[\begin{array}{c} (\mathbf{CA}) \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot (\mathbf{CA})^T + \mathbf{C} \cdot \mathbb{E} \{ \mathbf{w}_k \mathbf{w}_k^T \} \cdot \mathbf{C}^T + \mathbb{E} \{ \mathbf{v}_{k+1} \mathbf{v}_{k+1}^T \} \\ \mathbf{C} \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot (\mathbf{CA})^T \\ (\mathbf{CA}) \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot \mathbf{C}^T \\ \mathbf{C} \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot \mathbf{C}^T + \mathbb{E} \{ \mathbf{v}_k \mathbf{v}_k^T \} \end{array} \right] . \end{aligned} \quad (6)$$

As the set of equations (6) depends on the current estimation uncertainty, expressed by the respective covariances, the gains $\mathbf{N}$ and $\mathbf{L}$ are time varying and need to be updated in each discretization time instant. The remaining matrix $\mathbf{T}$ results from the above-mentioned equality constraint $\mathbf{T} = \mathbf{I} - \mathbf{NC}$.

To make this optimization result applicable for set-based estimators, the covariance matrices in (6) are replaced by shape matrices of ellipsoids that enclose a predefined confidence level, see also [4].

Under the assumption of the regularity of the matrix on the right hand side of (6), being ensured for standard detectability assumptions of a Kalman filter, the observer gains can be computed in closed analytic form. This is a big advantage over the classical TNL parameterization, where the parameterization results from the numerical solution of linear matrix inequalities.

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A homoclinic connection to a vertical Lyapunov orbit in the Sun-Jupiter system: a computer-assisted proof

Jakub Czwórnoć

AGH University of Krakow, Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Kraków, Poland
jczwornog@agh.edu.pl

Keywords: interval methods, homoclinic orbits, reversing symmetry, Circular Restricted Three-Body Problem, Brouwer degree

Introduction

Homoclinic and heteroclinic orbits to equilibria and periodic solutions arise when their unstable and stable manifolds intersect. They play an important role in spacecraft mission design [7]. For example, the heteroclinic connection between L_1 and L_2 was used in the Genesis mission [3] and ARTEMIS [2].

We present a computer-assisted proof of a non-transversal homoclinic connection to a vertical Lyapunov orbit in the Circular Restricted Three-Body Problem (CR3BP) for the Sun-Jupiter system with mass parameter $\mu = 9.5388114032796904 \cdot 10^{-4}$. The proof exploits the reversing symmetry of the system and uses a topological argument based on the Brouwer degree. All rigorous computations are performed in the CAPD library [4]. In particular, CAPD is used to perform a rigorous enclosure of the Poincaré map [5].

Vertical Lyapunov orbits

The equations of motion of the CR3BP with mass parameter μ admit a first integral, the Jacobi constant C_μ , and two symmetries: a reversing

symmetry R and a symmetry S of the form

$$\begin{aligned} R(x, y, z, \dot{x}, \dot{y}, \dot{z}) &= (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}) \\ S(x, y, z, \dot{x}, \dot{y}, \dot{z}) &= (x, y, -z, \dot{x}, \dot{y}, -\dot{z}). \end{aligned}$$

Vertical Lyapunov orbits near libration points $\{L_i\}_{i=1,2,3}$ are double symmetric with respect to S and R . Using these symmetries, finding such an orbit at the chosen energy value e of C_μ reduces to finding an initial condition of the form $v_e = (x_e, 0, z_e, 0, \dot{y}_e, 0) \in \text{Fix}(R)$, which is a solution to the following parameterized equation

$$\begin{cases} \pi_{(y,\dot{x})}\mathcal{P}_z(x, 0, z, 0, \dot{y}, 0) = 0 \\ C_\mu(x, 0, z, 0, \dot{y}, 0) - e = 0. \end{cases}$$

The existence of a smooth family $e \mapsto v_e$ over an explicit energy interval E is verified using the Interval Newton Method [6] for parameterized equations. Moreover, $v_e \in \text{Fix}(R)$ is a fixed point of the fourth iterate of the Poincaré map on $\Pi_y = \{y = 0\}$:

$$F(v_e) = \mathcal{P}_y^4(v_e) = v_e.$$

Homoclinic connection

Since the system is reversible and $v_e \in \text{Fix}(R)$, any point $w \in W^u(v_e, F) \cap \text{Fix}(R)$ gives rise to a homoclinic connection to the vertical orbit. This reduces to finding a solution of

$$\pi_{(\dot{x}, \dot{z})}\mathcal{P}_y\mathcal{P}_x(w) = 0 \quad w \in W^u(v_e, F). \quad (1)$$

The spectrum of $A = DF(v_e)$ is of the form $\{\lambda, \lambda^{-1}, a \pm bi, 1\}$ with $\lambda > 1$ and $a^2 + b^2 = 1$. Hence the unstable manifold $W^u(v_e, F)$ is one-dimensional. We work with the Poincaré map on the section $\Pi_y = \{y = 0\}$, so the dynamics is described by a 5-dimensional map. After removing the neutral direction via the Jacobi constant and a change of basis L , diagonalizing A , the equation (1) becomes a system of two equations with two variables (τ, e) , where τ parameterizes $W^u(v_e, F)$

$$G(\tau, e) = \pi_{(\dot{x}, \dot{z})}\mathcal{P}_y\mathcal{P}_x(\tau, w_e^u(\tau)) = 0.$$

Rigorous estimate of unstable manifold and Brouwer degree argument

The local unstable manifold $W^u(v_e, F)$ is rigorously estimated using cone conditions [1]. In the coordinate system induced by L , we verify that the map F satisfies cone conditions for two quadratic forms Q_h, Q_v over an explicit neighborhood of the fixed point via interval enclosure of its derivative. This gives a Lipschitz function $w_e^u(\tau)$ whose graph is locally $W^u(v_e, F)$, together with explicit bounds used to estimate G .

To prove the existence of a zero of G we use the Brouwer degree. Given a numerical approximation $\tilde{\zeta} = (\tilde{\tau}, \tilde{e})$ and an explicit box $U \subset \mathbb{R}^2$ around $\tilde{\zeta}$, we form the homotopy

$$H(t, \zeta) = tB(\zeta - \tilde{\zeta}) + (1 - t)G(\zeta), \quad t \in [0, 1],$$

where B is a linear isomorphism. If $0 \notin H(t, \partial U)$ for every $t \in [0, 1]$, then $\deg(G, U, 0) \neq 0$ and G has a zero in U .

Main result

Theorem 1. *Let*

$$\begin{aligned} \mu &= 9.5388114032796904 \cdot 10^{-4} \\ \tilde{e} &= 3.0268216642156945825 \end{aligned}$$

and

$$E = \tilde{e} + 10^{-8} \cdot [-1, 1].$$

In the CR3BP with fixed parameter μ , there exists $e \in E$ such that $C_\mu^{-1}(\{e\})$ contains a vertical Lyapunov orbit near L_1 admitting a homoclinic connection.

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Implicit Taylor Series Method for Delay Differential Equations

Anna Gierzkiewicz, Jakub Kural, Robert Szczelina

Jagiellonian University
{anna.gierzkiewicz,robert.szczelina}@uj.edu.pl
jakub.kural@student.uj.edu.pl

Keywords: Mathematics, Computer Science, Interval Methods, Delay Differential Equations, Implicit Taylor Method

We present an implicit Taylor method algorithm for rigorous integration of delay differential equations. It is a modification to algorithms considered in [1] and [2] which use explicit Taylor method to compute rigorous bounds on a set of solutions to delay differential equations of the form

$$\begin{cases} x'(t) = f(x(t), x(t - \tau_1), \dots, x(t - \tau_m)) & t \geq 0 \\ x(t) = \psi(t) & t \in [-\tau, 0] \end{cases}$$

where $\tau = \tau_1 \leq \dots \leq \tau_m \leq 0$.

We extend them by additionally using implicit Taylor method together with interval Newton method to further improve the bounds on the solutions. It is especially important when studying stiff equations, where the explicit method might fail to converge and the errors can accumulate quickly.

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Existence of a periodic orbit in pseudospectral projection of Cubic Ikeda

Darina Serogina

Jagiellonian University
darina.serogina@student.uj.edu.pl

Keywords: Interval Methods, Delayed Differential Equations, Covering Relation, Computer-Assisted Proofs

The system

This talk will focus on analysing a dynamical system given by a specific delayed differential equation (DDE). The family of such equations is of the following form:

$$x'(t) = f(x(t), x(t - \tau)), \quad (1)$$

$$x(t) = \theta(t) \quad t \in [-\tau, 0] \quad (2)$$

where $\tau > 0$ is a given parameter (the delay) and θ is an initial function.

The equation of interest is the Cubic Ikeda equation:

$$x'(t) = a \cdot (x(t - \tau) - x(t - \tau)^3), \quad x, \tau \in \mathbb{R}. \quad (3)$$

There exist methods of rigorous computations for general systems of DDEs (see [1]), and they have been used to prove the existence of periodic orbits in equations of the form (1), however they cannot yet be used for proving more complicated kinds of dynamical phenomena, like chaos. Thus our goal is to find an easier model that will approximate the dynamics and see whether we can succeed in proving chaos there. The hope is to recreate them later on the original system.

In this work we used a *pseudospectral projection* for approximating the DDE by a system of ODEs:

$$x'(t) = a \cdot (x(t - \tau) - x(t - \tau)^3) \rightsquigarrow \begin{cases} x'[0] = a \cdot (x[N] - x[N]^3) \\ x'[1] = M_1 \cdot x[1] \\ \vdots \quad \quad \quad \vdots \\ x'[N] = M_N \cdot x[N] \end{cases} \quad (4)$$

where M is a $(N + 1) \times (N + 1)$ dimensional matrix, and M_i is the i -th row of this matrix, for convenience counting from 0.

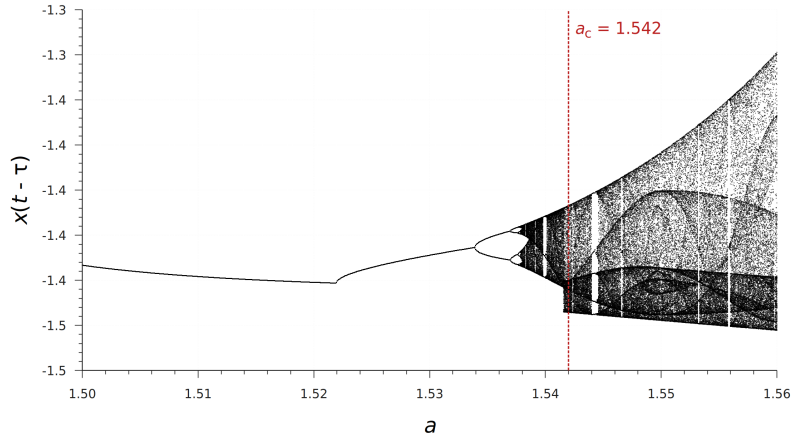


Figure 1: *Bifurcation diagram for parameter $a \in [1.5, 1.56]$ with $a = 1.542$ being marked with red line.*

In this new system given by (4) the first step is to prove existence of a periodic orbit, which is apparently unstable. Later on our goal will be to find an orbit homoclinic to that orbit, and this construction can be used to guide a proof of the symbolic dynamics in the system.

Since the main goal is to later prove chaos (symbolic dynamics) in the system, we will not be using interval Newton's method to prove the existence of the periodic solution. Instead the idea is to prove it by using covering relations (see [2]) for a Poincaré map on a suitable section. The rigorous enclosure of the images of this Poincaré map

will be constructed using CAPD in accordance with [3], and those estimates will be used to obtain a computer assisted proof of some covering relations.

Main results

Theorem 1. *For parameter values $a = 1.542$ and $\tau = 1$ in the dynamical system given by (4) there exists a periodic orbit with period 1.*

Proof. We construct a h -set N that satisfies $N \xrightarrow{P} N$ for P being a Poincaré map in this system. Checking the covering relation is done by using the CAPD library for calculating an enclosure of the image of N through the map P . The orbit (apparently) has one unstable direction, while the rest appear to be stable. The behaviour under the Poincaré map and the covering properties can be seen in Figure 2. $\square$

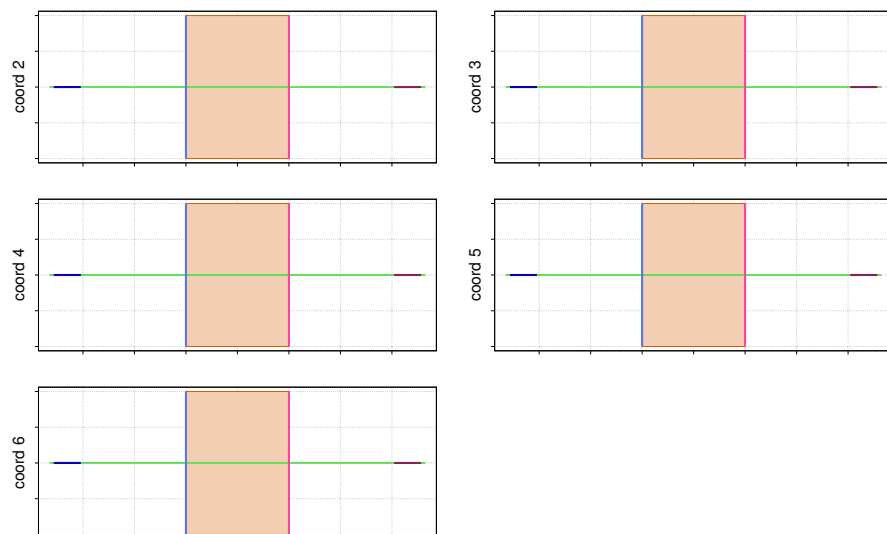


Figure 2: *Covering relation of the h -set N (orange) under the Poincaré map (green). The horizontal axis is the first, apparently unstable, coordinate. Blue and pink lines indicate the exit boundary N_c^- (lighter) and its image (darker).*

We note that this is just the preliminary result. The plan for the nearest future is to prove the following:

Conjecture 1. *There exists a homoclinic orbit to the periodic orbit given in Theorem 1.*

The insight into the dynamics of the ODE (4) will then be used to guide future research into proving similar results for the full DDE (3).

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