

A Comparison between the Theorem of Ehlich and Zeller and Polytopic Uncertainty Representations for the Development of Interval Observers

Andreas Rauh¹, Fabian Schneider², Marit Lahme¹, and
Christian Wölfel²

¹ Carl von Ossietzky Universität Oldenburg
Distributed Control in Interconnected Systems
D-26111 Oldenburg, Germany
{Andreas.Rauh, Marit.Lahme}@uni-oldenburg.de

² Ruhr University Bochum
Chair of Automation
D-44780 Bochum, Germany
{Fabian.Schneider, Christian.Woelfel}@atp.rub.de

Keywords: Interval observer design, Nonlinear systems, Polytopic uncertainty representation, Theorem of Ehlich and Zeller

Interval Observer Design for Linear Systems

Interval observers provide a means to perform state estimation in a bounded error framework, where lower and upper bounding trajectories are determined so that the sets of reachable states are enclosed at each point in time despite bounded uncertainty in the dynamic system model as well as in the measurement process [2, 5].

Originally, this design was developed for linear time-invariant (LTI) system models

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \mathbf{w}(t) \quad (1)$$

$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{v}(t) \quad (2)$$

Here, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ the control vector, and $\mathbf{y}(t) \in \mathbb{R}^q$ the vector of measurements. In addition, $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are bounded disturbances in the form $\mathbf{w}(t) \in [\underline{\mathbf{w}}(t) ; \overline{\mathbf{w}}(t)]$ and $\mathbf{v}(t) \in [\underline{\mathbf{v}}(t) ; \overline{\mathbf{v}}(t)]$, respectively.

In the original publications on interval observer design for LTI systems (1) with the uncertain initial conditions $\mathbf{x}(0) \in [\underline{\hat{\mathbf{x}}}(t) ; \overline{\hat{\mathbf{x}}}(t)]$, the observer was defined as

$$\dot{\underline{\hat{\mathbf{x}}}}(t) = \mathbf{A} \cdot \underline{\hat{\mathbf{x}}}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \underline{\mathbf{w}}(t) + \mathbf{H} \cdot (\mathbf{y}_m(t) + \underline{\mathbf{v}}(t) - \mathbf{C} \cdot \underline{\hat{\mathbf{x}}}(t)) \quad (3)$$

$$\dot{\overline{\hat{\mathbf{x}}}}(t) = \mathbf{A} \cdot \overline{\hat{\mathbf{x}}}(t) + \mathbf{B} \cdot \mathbf{u}(t) + \overline{\mathbf{w}}(t) + \mathbf{H} \cdot (\mathbf{y}_m(t) + \overline{\mathbf{v}}(t) - \mathbf{C} \cdot \overline{\hat{\mathbf{x}}}(t)) \quad (4)$$

with the sensor measurements $\mathbf{y}_m(t)$. For decoupling the lower and upper bounding trajectories of the estimator according to (3) and (4), it is typically assumed that the lower state bound $\underline{\hat{\mathbf{x}}}(t)$ and $\mathbf{B} \cdot \mathbf{u}(t)$ are element-wise non-negative and that all off-diagonal elements of $\mathbf{A} - \mathbf{H}\mathbf{C}$ are non-negative, i.e., this matrix is Metzler. Moreover, stability needs to be ensured by a suitable choice of $\mathbf{H}$.

In recent years, generalizations were published for systems with bounded uncertainty in the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ and for systems with negative bounds $\underline{\hat{\mathbf{x}}}(t)$. Moreover, relaxations of the Metzler property of $\mathbf{A} - \mathbf{H}\mathbf{C}$ become possible by either coordinate transformations [4] or by extending the observer differential equations (3) and (4) by the so-called TNL approach, cf. [7].

Generalization to Nonlinear System Models

For nonlinear system models, a polytopic (or norm-bounded) uncertainty model is typically determined to enclose the variability of all state-dependent dynamics after re-writing the system model in the quasi-linear representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t)) \cdot \mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t)) \cdot \mathbf{u}(t) + \mathbf{w}(t) \quad (5)$$

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{x}(t)) \cdot \mathbf{x}(t) + \mathbf{v}(t) \quad (6)$$

As discussed in [6], such kind of polytopic uncertainty representation of the state-dependent matrices in the quasi-linear model (5)

easily leads to enclosing many combinations of values that are not at all attainable during the evolution of the system.

As shown in [1], the theorem of Ehlich and Zeller [3] — that was originally developed to determined range enclosures for polynomial expressions over a finite range of their input argument(s) — can be exploited as an alternative.

Consider a univariate polynomial

$$p(x) = \sum_{i=0}^d a_i x^i \quad (7)$$

of degree d with $x \in [\mathcal{X}] = [\underline{x} ; \bar{x}]$ and determine the Chebychev points $x^{(i)} = \frac{\underline{x} + \bar{x}}{2} + \frac{\bar{x} - \underline{x}}{2} \cos\left(\frac{(2i-1)\pi}{2N}\right)$, where $N > d$, $N \in \mathbb{N}$. Then, the bounds

$$\begin{aligned} \sup_{x \in [\mathcal{X}]} p(x) &\leq \frac{1}{2} \left((C_p + 1) p_{\max}^{\mathcal{X}^{(N)}} - (C_p - 1) p_{\min}^{\mathcal{X}^{(N)}} \right) \quad \text{and} \\ \inf_{x \in [\mathcal{X}]} p(x) &\geq \frac{1}{2} \left((C_p + 1) p_{\min}^{\mathcal{X}^{(N)}} - (C_p - 1) p_{\max}^{\mathcal{X}^{(N)}} \right) \end{aligned} \quad (8)$$

with

$$\begin{aligned} p_{\min}^{\mathcal{X}^{(N)}} &= \min_{i \in \{1, \dots, N\}} p\left(x^{(i)}\right) \quad \text{and} \\ p_{\max}^{\mathcal{X}^{(N)}} &= \max_{i \in \{1, \dots, N\}} p\left(x^{(i)}\right) , \end{aligned} \quad (9)$$

where

$$C_p = \left(\cos\left(\frac{d\pi}{2N}\right) \right)^{-1} \quad (10)$$

enclose the complete range of the function $p(x)$ over its input domain.

In the frame of an interval observer synthesis, the benefit of this approach is the ability to better capture correlations that exist between the elements of the state-dependent matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{C}(\mathbf{x})$. In this contribution, we investigate this procedure not only for univariate polynomials but also for the multi-variate case and thoroughly analyze the influence of this type of range computation over the classically employed polytopic approach.

References

- [1] R. Dehnert, M. Rosik, S. Lerch, A. Rauh, and B. Tibken. Robust observer-based control design for stochastic discrete-time quasilinear systems via LMIs. *IEEE Transactions on Automatic Control*, 70(9):5660–5674, 2025. DOI: 10.1109/TAC.2025.3545515.
- [2] D. Efimov, T. Raïssi, S. Chebotarev, and A. Zolghadri. Interval state observer for nonlinear time varying systems. *Automatica*, 49(1):200–205, 2013.
- [3] H. Ehlich and K. Zeller. Schwankung von Polynomen zwischen Gitterpunkten. *Mathematische Zeitschrift*, 86(1):41–44, 1964. In German.
- [4] J. Kersten, A. Rauh, and H. Aschemann. State-space transformations of uncertain systems with purely real and conjugate-complex eigenvalues into a cooperative form. In *Proceedings of 23rd International Conference on Methods and Models in Automation and Robotics 2018*, Międzyzdroje, Poland, 2018.
- [5] T. Raïssi, D. Efimov, and A. Zolghadri. Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57:260–265, 2012.
- [6] A. Rauh and M. Lahme. The theorem of ehlich and zeller to develop polytopic models for set-based observers. In *Summer Workshop on Interval Methods (SWIM)*, Rennes, France, 2025. Online available via DOI:10.13140/RG.2.2.33857.47207.
- [7] Z. Wang, C.-C. Lim, and Y. Shen. Interval observer design for uncertain discrete-time linear systems. *Systems & Control Letters*, 116:41–46, 2018.