

Parameterization Approaches for Luenberger-Type Contractors in Set-Based State Estimation

Andreas Rauh and Marit Lahme

Carl von Ossietzky Universität Oldenburg
Distributed Control in Interconnected Systems
D-26111 Oldenburg, Germany

`{Andreas.Rauh, Marit.Lahme}@uni-oldenburg.de`

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Approaches for Bounded Error State Estimation

Typically, set-based state estimation approaches for dynamic systems with bounded uncertainty are classified into predictor–corrector approaches [6] and interval observers [3]. Predictor–corrector approaches rely on forecasting reachable domains in the state-space using the system’s open-loop dynamics until the time instant at which measured data become available. Then, the predicted sets are contracted with the help of measurement information which is also assumed to be subject to bounded uncertainty. As shown in [1, 5], this contraction procedure can be replaced by a Luenberger-like observer approach. Computationally efficient implementations can be achieved if the sets of reachable states are represented with the help of zonotopes. In contrast, interval observers commonly assume the direct computation of lower and upper bounding trajectories for each of the state variables. Due to the required assumptions on monotonicity of the error dynamics, the design conditions are subject to stronger constraints than in the case of Luenberger-type contractors.

Restrictions of Interval Observer Synthesis

To relax the restrictions that hold due to monotonicity properties [3], approaches relying on coordinate transformations have been introduced. In general, these approaches transform a non-cooperative realization into a cooperative one [2]. Alternatively, extensions of the observer structure by further design degrees of freedom [7] have been developed which do not only account for measurements at a single point in time but which add two further gain matrices and a measurement at an additional time instant. Typically, these approaches are parameterized with the help of linear matrix inequality techniques, where optimality of the solution is achieved by a minimization of the worst-case amplification within the signal path from the uncertain external influence factors to the estimated state enclosures.

In this paper, we replace the design criteria for this so-called TNL observer technique (named after the three design matrices) by an optimization that is inspired by the minimization of the one time-step estimation error co-variance as it is done in the frame of the design of Kalman filters for linear dynamic systems. Generalizing this approach to the TNL observer according to the following strategy allows for more easily weighting the magnitudes in the individual uncertain influence factors in the process and measurement models. However, cooperativity is not necessarily obtained so that the implementation is then performed with the help of a zonotopic state contractor. To get an impression when this type of parameterization is advantageous in comparison to the classical TNL design wrt. existence of solutions and tightness of the computed state enclosures, different numerical case studies are presented on the basis of system models that are inspired by real-life applications.

Sketch of the Gain Optimization Procedure

Assume a discrete-time state estimator given in the form [7]

$$\hat{\mathbf{x}}_{k+1} = \mathbf{T}\mathbf{A}\hat{\mathbf{x}}_k + \mathbf{T}\mathbf{B}\mathbf{u}_k + \mathbf{L} \cdot (\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_k) + \mathbf{\Delta}_k + \mathbf{N}\mathbf{y}_{k+1} \quad (1)$$

with the state estimate $\hat{\mathbf{x}}_k$, the control signal $\mathbf{u}_k$, the measurement $\mathbf{y}_k$, and the disturbances Δ_k resulting from process and measurement noise. Assume further that $\mathbf{T} = \mathbf{I} - \mathbf{N}\mathbf{C}$ holds, where $\mathbf{I}$ is the identity matrix of appropriate dimensions, and $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{L}$ are the design degrees of freedom of the observer, cf. [7].

In contrast to the classical parameterization of this observer, we do not aim at minimizing an ∞ -norm criterion but rather optimize a quadratic norm on the estimation error dynamics given by

$$\mathbf{e}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k, \quad (2)$$

where $\mathbf{x}_k$ is the true, however, not directly measurable system state. Using the ansatz (1) and assuming that the dynamics matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are accurately known, the error dynamics are then given by

$$\begin{aligned} \mathbf{e}_{k+1} &= (\mathbf{T}\mathbf{A} - \mathbf{L}\mathbf{C}) \mathbf{e}_k + \Delta_k \\ &= (\mathbf{A} - \mathbf{N}\mathbf{C}\mathbf{A} - \mathbf{L}\mathbf{C}) \mathbf{e}_k + \Delta_k. \end{aligned} \quad (3)$$

If the uncertainties for state reconstruction and external perturbations are assumed to be normally distributed, an optimal observer parameterization is obtained by the minimization of the one-time-step error covariance

$$\begin{aligned} \mathbb{E} \{ \mathbf{e}_{k+1} \mathbf{e}_{k+1}^T \} &= (\mathbf{A} - \mathbf{N}\mathbf{C}\mathbf{A} - \mathbf{L}\mathbf{C}) \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot (\mathbf{A} - \mathbf{N}\mathbf{C}\mathbf{A} - \mathbf{L}\mathbf{C})^T \\ &\quad + (\mathbf{I} - \mathbf{N}\mathbf{C}) \mathbf{E} \cdot \mathbb{E} \{ \mathbf{w}_k \mathbf{w}_k^T \} \cdot \mathbf{E}^T (\mathbf{I} - \mathbf{N}\mathbf{C})^T \\ &\quad + \begin{bmatrix} \mathbf{L} & \mathbf{N} \end{bmatrix} \cdot \begin{bmatrix} \mathbb{E} \{ \mathbf{v}_k \mathbf{v}_k^T \} & \mathbf{0} \\ \mathbf{0} & \mathbb{E} \{ \mathbf{v}_{k+1} \mathbf{v}_{k+1}^T \} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{L}^T \\ \mathbf{N}^T \end{bmatrix}. \end{aligned} \quad (4)$$

Here, the term $\mathbf{E}\mathbf{w}_k$ represents the influence of the process uncertainty $\mathbf{w}_k$ on the state update $\mathbf{x}_{k+1}$; $\mathbb{E} \{ \mathbf{w}_k \mathbf{w}_k^T \}$ is the process noise covariance; $\mathbb{E} \{ \mathbf{v}_k \mathbf{v}_k^T \}$ and $\mathbb{E} \{ \mathbf{v}_{k+1} \mathbf{v}_{k+1}^T \}$ are the measurement noise covariances related to additive uncertainty of the measurements $\mathbf{y}_k$ and $\mathbf{y}_{k+1}$, respectively.

Now, an optimization of the gain matrices $\mathbf{L}$ and $\mathbf{N}$ becomes possible by a minimization of the trace

$$\text{tr} \left(\mathbb{E} \{ \mathbf{e}_{k+1} \mathbf{e}_{k+1}^T \} \right) \quad (5)$$

of the estimation error covariance. The necessary optimality conditions result from setting the respective partial derivatives with respect to $\mathbf{L}$ and $\mathbf{N}$ to zero, leading to the linear set of equations

$$\begin{aligned} & \left[\mathbf{A} \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot (\mathbf{CA})^T + \mathbf{E} \cdot \mathbb{E} \{ \mathbf{w}_k \mathbf{w}_k^T \} \cdot \mathbf{C}^T \mid \mathbf{A} \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot \mathbf{C}^T \right] \\ &= [\mathbf{N} \quad \mathbf{L}] \cdot \\ & \left[\begin{array}{c} (\mathbf{CA}) \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot (\mathbf{CA})^T + \mathbf{C} \cdot \mathbb{E} \{ \mathbf{w}_k \mathbf{w}_k^T \} \cdot \mathbf{C}^T + \mathbb{E} \{ \mathbf{v}_{k+1} \mathbf{v}_{k+1}^T \} \\ \mathbf{C} \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot (\mathbf{CA})^T \\ (\mathbf{CA}) \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot \mathbf{C}^T \\ \mathbf{C} \cdot \mathbb{E} \{ \mathbf{e}_k \mathbf{e}_k^T \} \cdot \mathbf{C}^T + \mathbb{E} \{ \mathbf{v}_k \mathbf{v}_k^T \} \end{array} \right] . \end{aligned} \quad (6)$$

As the set of equations (6) depends on the current estimation uncertainty, expressed by the respective covariances, the gains $\mathbf{N}$ and $\mathbf{L}$ are time varying and need to be updated in each discretization time instant. The remaining matrix $\mathbf{T}$ results from the above-mentioned equality constraint $\mathbf{T} = \mathbf{I} - \mathbf{NC}$.

To make this optimization result applicable for set-based estimators, the covariance matrices in (6) are replaced by shape matrices of ellipsoids that enclose a predefined confidence level, see also [4].

Under the assumption of the regularity of the matrix on the right hand side of (6), being ensured for standard detectability assumptions of a Kalman filter, the observer gains can be computed in closed analytic form. This is a big advantage over the classical TNL parameterization, where the parameterization results from the numerical solution of linear matrix inequalities.

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