Characterizing indistinguishability sets in case of outliers

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Indistinguishability set

In the context of set-membership estimation [5], we have a measurement vector $\mathbf{y}$ and we know that this vector is related to some unknown vector of interest $\mathbf{x}$ by the relation

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) + \mathbf{e} \tag{1}$$

where $\mathbf{e} \in \mathbb{E}$ is an unknown but bounded vector and $\mathbf{f}$ is a known function. The set of all feasible solutions is

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}) \tag{2}$$

where $\mathbb{Y} = \mathbf{y} - \mathbb{E}$.

We define the indistinguishability set $\hat{\mathbb{X}}(\mathbf{x})$ associated to $\mathbf{x}$ as the set of all $\hat{\mathbf{x}}$ that could produce the same measurement vector as $\mathbf{x}$. We have

$$\begin{array}{rcc} \hat{\mathbb{X}}(\mathbf{x}) & = & \{\hat{\mathbf{x}} \mid \exists \mathbf{e} \in \mathbb{E}, \exists \hat{\mathbf{e}} \in \mathbb{E}, \mathbf{f}(\mathbf{x}) + \mathbf{e} = \mathbf{f}(\hat{\mathbf{x}}) + \hat{\mathbf{e}}\} \\ & = & \{\hat{\mathbf{x}} \mid \exists \mathbf{e} \in \mathbb{E}, \exists \hat{\mathbf{e}} \in \mathbb{E}, \mathbf{f}(\hat{\mathbf{x}}) = \mathbf{f}(\mathbf{x}) + \mathbf{e} - \hat{\mathbf{e}}\} \\ & = & \{\hat{\mathbf{x}} \mid \mathbf{f}(\hat{\mathbf{x}}) \in \mathbf{f}(\mathbf{x}) + \mathbb{E} \ominus \mathbb{E}\} \\ & = & \mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}) + \mathbb{E} \ominus \mathbb{E}) \end{array} \tag{3}$$

where $\ominus$ is the Minkowski difference. As a consequence, computing $\hat{\mathbb{X}}(\mathbf{x})$ corresponds to a set inversion problem. An algorithm such as SIVIA [4] can be used to compute an inner and an outer approximation of $\hat{\mathbb{X}}(\mathbf{x})$.

We would like to make this approach more robust by considering the case where outliers exist.

In case of outliers

In case of outliers [3], we generally assume that the set $\mathbb{E} = [e_1] \times \cdots \times [e_m]$ is a centered box and we reformulate the set inversion problem (2) as:

$$\mathbb{X} = \bigcap_{i}^{\{q\}} f_i^{-1}([y_i]) \tag{4}$$

where $[y_i] = y_i - [e_i]$ and q is the Maximum Number of Outliers (MNO). Using the transformation provided by (3) to get a set inversion definition for the indistinguishability set $\hat{\mathbb{X}}(\mathbf{x})$ is not possible yet. We need first to express (4) as the inverse of a set.

For this, let us introduce the pseudo-norm $\mathcal{L}_0$ used as a sparsity measure:

$$\mathcal{L}_0 : \mathbf{s} \mapsto \|\mathbf{s}\|_0 = card(\{i | s_i \neq 0\}) \tag{5}$$

Proposition 1. *Define the sparse set*

$$\mathbb{S}_q = \{\mathbf{s} \in \mathbb{R}^n \mid \|\mathbf{s}\|_0 \leq q\}$$

We have:

$$\bigcap_{i}^{\{q\}} f_i^{-1}([y_i]) = \mathbf{f}^{-1}(\mathbf{y} + \mathbb{E} \oplus \mathbb{S}_q)$$

Proof. The MNO assumption induces: $\exists \mathbf{s} \in \mathbb{S}_q, \mathbf{y} = \mathbf{f}(\mathbf{x}) + \mathbf{e} + \mathbf{s}$ $\square$

Proposition 2. *The indistinguishability set* $\hat{\mathbb{X}}(\mathbf{x})$ *can be expressed as the set inversion problem*

$$\hat{\mathbb{X}}(\mathbf{x}) = \mathbf{f}^{-1}(\mathbf{f}(\mathbf{x}) + 2\mathbb{E} \oplus \mathbb{S}_{2q})$$

Proof. The proof uses the triangular inequality of the pseudo-norm $\mathcal{L}_0$ which yields the equality $\mathbb{S}_q \oplus \mathbb{S}_q = \mathbb{S}_{2q}$. $\square$

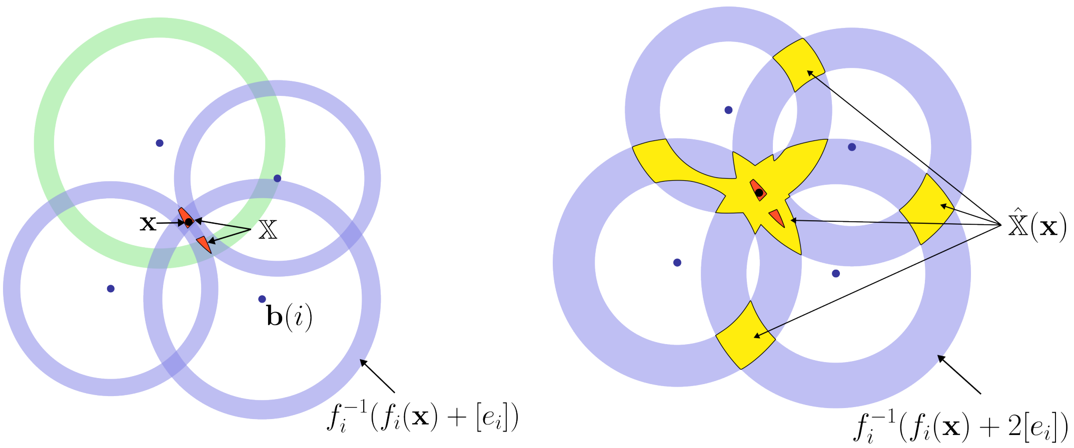
Corollary 3. *We have*

$$\hat{\mathbb{X}}(\mathbf{x}) = \bigcap_{i}^{\{2q\}} f_i^{-1}(f_i(\mathbf{x}) + 2\mathbb{E})$$

This formula allows us to use an efficient contractor approach [2][1] with the relaxed intersection operators for contractors.

Application

We apply the previous results to the 2-D localization of a point using distance measurements to multiple beacons, as illustrated below with 4 beacons and a MNO $q = 1$.



Left: 4 distance measurements generated by $\mathbf{x}$, the one in green (upper-left) being an outlier. In red the set inversion solution $\mathbb{X}$.
Right: The indistinguishability set $\hat{\mathbb{X}}(\mathbf{x})$ computed from the 4 true distances from $\mathbf{x}$ to the beacons $\mathbf{b}(i)$, taking into account the doubled bounded error and the MNO $q = 1$.

The position vector is

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T \quad (6)$$

And the observation functions are

$$f_i : \mathbf{x} \mapsto \sqrt{(x_1 - b_1(i))^2 + (x_2 - b_2(i))^2} \qquad (7)$$

with $\mathbf{b}(i) = (b_1(i), b_2(i))$ the known positions of the i-th beacon.

Using contractors for the n distance constraints and the relaxed intersection, $\hat{\mathbb{X}}(\mathbf{x})$ is computed with different count of outliers and beacons in a simple interactive simulation.

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