

Global changes in the structure of maximal invariant sets

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Keywords: Attractor, Entropy, Interval Methods for ODEs

Introduction

Attracting invariant sets are easily observable in numerical simulation. On the other hand, understanding internal dynamics on such invariant set is challenging. In particular, the observed maximal invariant may not be minimal. A typical situation is when an attracting cycle coexists with a chaotic set and a very long transition time is needed to observe the convergence to the periodic solution.

We present a computational framework for verification that a dissipative ODE admits a compact and connected maximal invariant set. We also provide a methodology to bound from below complexity of the dynamics – a lower bound on the topological entropy of the system is computed by means of interval-based methods. We also show, that local bifurcations of periodic orbits may lead to changes in global structure of the maximal invariant set enriching global orbit structure.

All main results are based on [4] and they are computer-assisted. We use the CAPD library [1] to compute bounds on the solutions to underlying ODEs, Poincaré maps [2] and their partial derivatives.

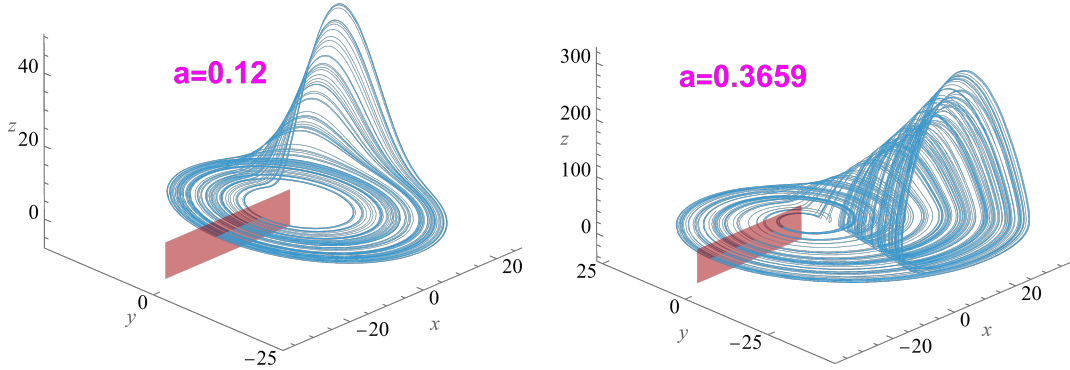


Figure 1: Poincaré section and a typical trajectory of (1) for $a = \min \mathcal{A}$ (left) and $a = \max \mathcal{A}$ (right).

Main results

Consider the Rössler system [3]

$$\dot{x} = -(y + z), \quad \dot{y} = x + ay, \quad \dot{z} = b + z(x - c) \quad (1)$$

with fixed $b = 0.2$ and $c = 15$. We analyse (1) for an explicit range of parameter values $a \in \mathcal{A} := [0.12, 0.3659]$. Denote by

$$\Pi := \{(x, y, z) \in \mathbb{R}^3 : y = 0 \wedge \dot{y} = x < 0\}, \quad P_a : \Pi \rightarrow \Pi$$

a Poincaré section and the corresponding Poincaré map, respectively. We will use (x, z) coordinates on Π . Numerical simulation shows that for a range of parameter values a large maximal invariant set of (1) exists and it changes its structure with the parameter – see Figure 1.

The following theorem summarizes results from [4] about the dynamics of (1).

Theorem 1. Put $\mathcal{T} := [-30.53, -3] \times [0.004, 0.011] \subset \Pi$.

1. For $a \in \mathcal{A}$, the Poincaré map P_a is well defined and smooth on $\mathcal{T}$ and $P_a(\mathcal{T}) \subset \text{int}\mathcal{T}$ – see Figure 2.
2. P_a undergoes a saddle-node bifurcation at the points (x_i^*, z_i^*, a_i^*) , $i = 1, \dots, 5$ with

$$|x_i^* - x_i| \leq 3 \cdot 10^{-11}, \quad |z_i^* - z_i| \leq 10^{-14}, \quad |a_i^* - a_i| \leq 6 \cdot 10^{-14},$$

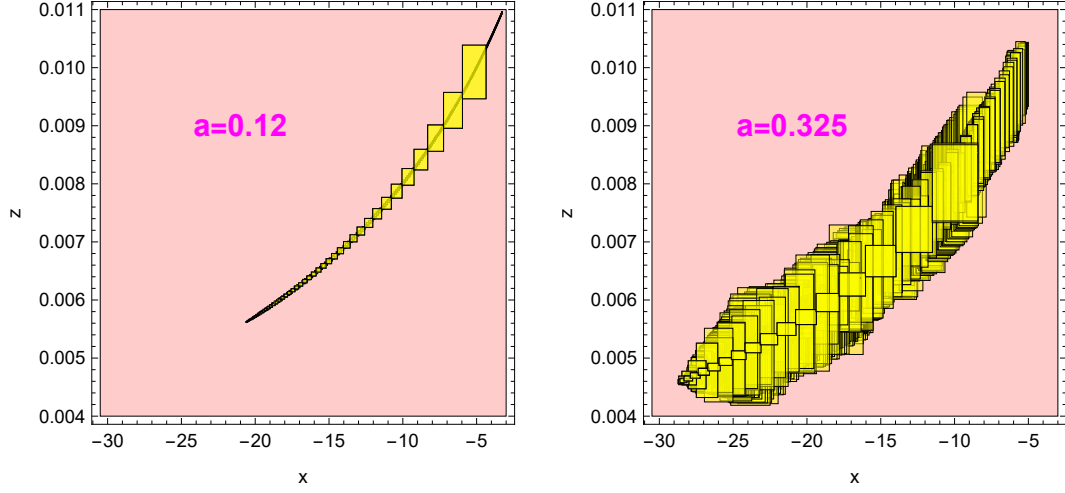


Figure 2: Trapping region $\mathcal{T}$ (in pink) and a bound on $P_a(\mathcal{T})$ (in yellow) for two selected parameter values.

where

i	x_i	z_i	a_i
1	-24.98615641824929	0.005003695953767296	0.2445890212249042
2	-27.90101006311546	0.004663547021688063	0.3119866509093180
3	-29.22211573599117	0.004524155280447693	0.3405236989996505
4	-29.89377365336397	0.004456435009829962	0.3546641965836549
5	-30.25817997684925	0.004420535075303823	0.3623180183723328

The branches of periodic points continue to exist at least to $a = \max \mathcal{A}$ – see Figure 3 (bottom).

3. For $a \in \mathcal{A}$ there is an invariant set $\mathcal{H}_a \subset \Pi$. The mapping $P_a|_{\mathcal{H}_a} : \mathcal{H}_a \rightarrow \mathcal{H}_a$ is semiconjugated to a subshift of finite type, with the number of symbols varying between 3 and 12. The topological entropy of $P_a|_{\mathcal{H}_a}$ is growing with the parameter reaching $h_{top}(P_{\max \mathcal{A}}) \geq \ln 12$ – see Figure 3 (top).

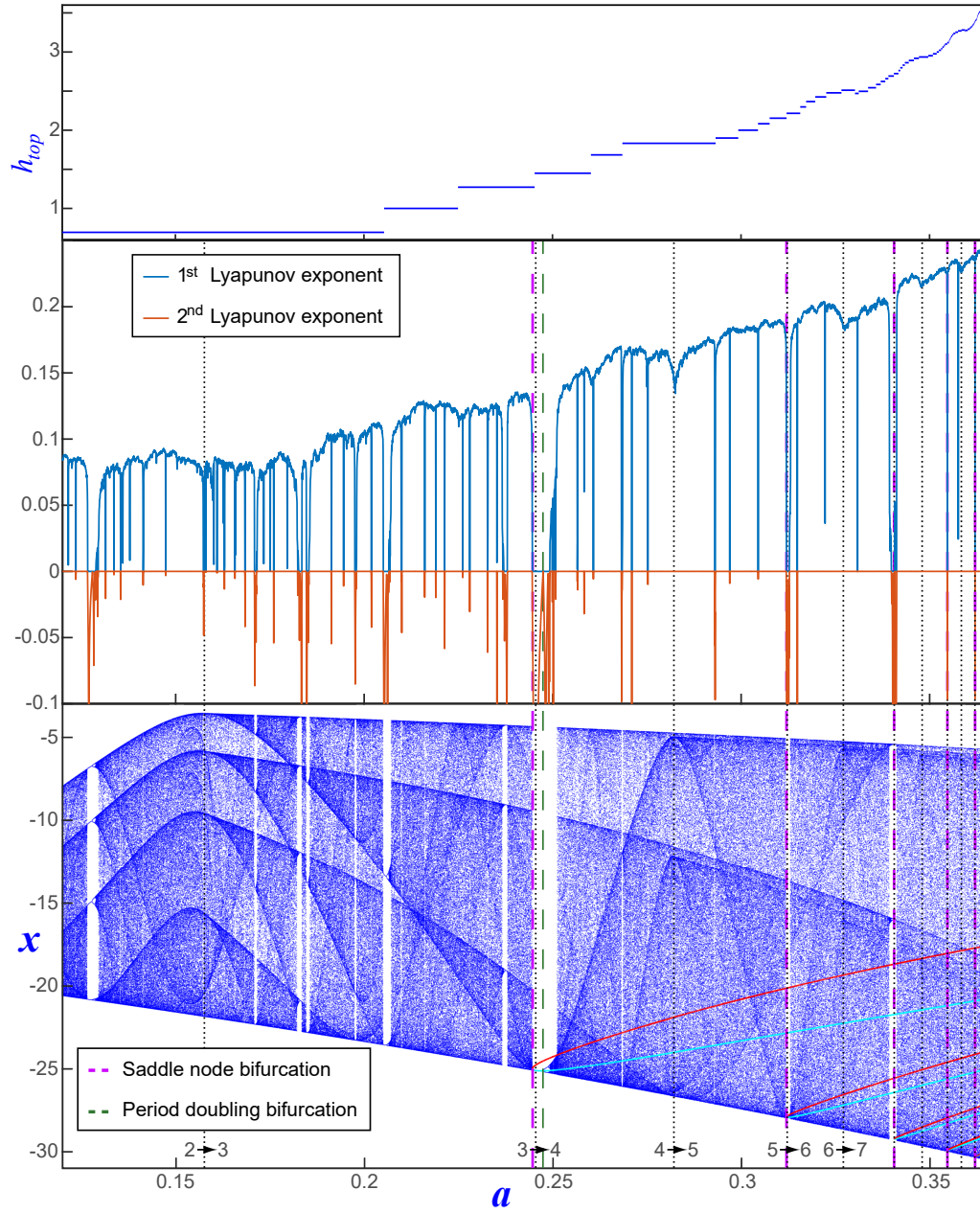


Figure 3: Top: lower bound on the topological entropy. Middle: Lyapunov exponents. Bottom: Bifurcation diagram representing the x -coordinate of the intersection points of the attractor with the Poincaré section Π . The dashed vertical segments indicate some saddle-node (in purple) and period-doubling (in green) bifurcations. Blue and red curves show stable and unstable families of periodic orbits born from the previous saddle-node bifurcations.

References

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