

Existence of a periodic orbit in pseudospectral projection of Cubic Ikeda

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The system

This talk will focus on analysing a dynamical system given by a specific delayed differential equation (DDE). The family of such equations is of the following form:

$$x'(t) = f(x(t), x(t - \tau)), \quad (1)$$

$$x(t) = \theta(t) \quad t \in [-\tau, 0] \quad (2)$$

where $\tau > 0$ is a given parameter (the delay) and θ is an initial function.

The equation of interest is the Cubic Ikeda equation:

$$x'(t) = a \cdot (x(t - \tau) - x(t - \tau)^3), \quad x, \tau \in \mathbb{R}. \quad (3)$$

There exist methods of rigorous computations for general systems of DDEs (see [1]), and they have been used to prove the existence of periodic orbits in equations of the form (1), however they cannot yet be used for proving more complicated kinds of dynamical phenomena, like chaos. Thus our goal is to find an easier model that will approximate the dynamics and see whether we can succeed in proving chaos there. The hope is to recreate them later on the original system.

In this work we used a *pseudospectral projection* for approximating the DDE by a system of ODEs:

$$x'(t) = a \cdot (x(t - \tau) - x(t - \tau)^3) \rightsquigarrow \begin{cases} x'[0] = a \cdot (x[N] - x[N]^3) \\ x'[1] = M_1 \cdot x[1] \\ \vdots \quad \quad \quad \vdots \\ x'[N] = M_N \cdot x[N] \end{cases} \quad (4)$$

where M is a $(N + 1) \times (N + 1)$ dimensional matrix, and M_i is the i -th row of this matrix, for convenience counting from 0.

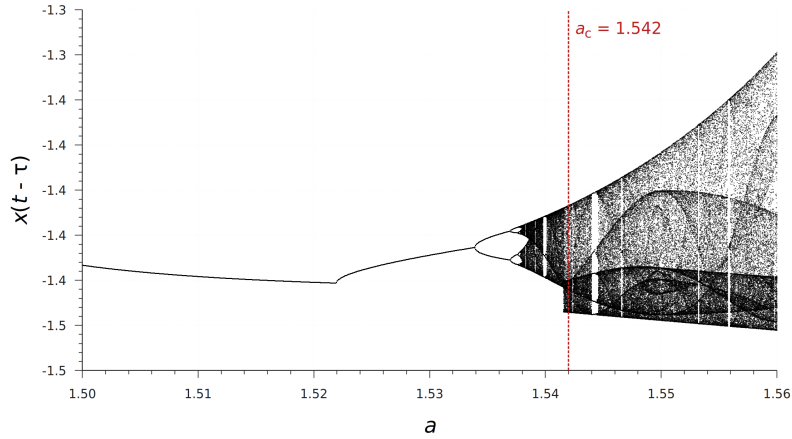


Figure 1: *Bifurcation diagram for parameter $a \in [1.5, 1.56]$ with $a = 1.542$ being marked with red line.*

In this new system given by (4) the first step is to prove existence of a periodic orbit, which is apparently unstable. Later on our goal will be to find an orbit homoclinic to that orbit, and this construction can be used to guide a proof of the symbolic dynamics in the system.

Since the main goal is to later prove chaos (symbolic dynamics) in the system, we will not be using interval Newton's method to prove the existence of the periodic solution. Instead the idea is to prove it by using covering relations (see [2]) for a Poincaré map on a suitable section. The rigorous enclosure of the images of this Poincaré map

will be constructed using CAPD in accordance with [3], and those estimates will be used to obtain a computer assisted proof of some covering relations.

Main results

Theorem 1. *For parameter values $a = 1.542$ and $\tau = 1$ in the dynamical system given by (4) there exists a periodic orbit with period 1.*

Proof. We construct a h -set N that satisfies $N \xrightarrow{P} N$ for P being a Poincaré map in this system. Checking the covering relation is done by using the CAPD library for calculating an enclosure of the image of N through the map P . The orbit (apparently) has one unstable direction, while the rest appear to be stable. The behaviour under the Poincaré map and the covering properties can be seen in Figure 2. $\square$

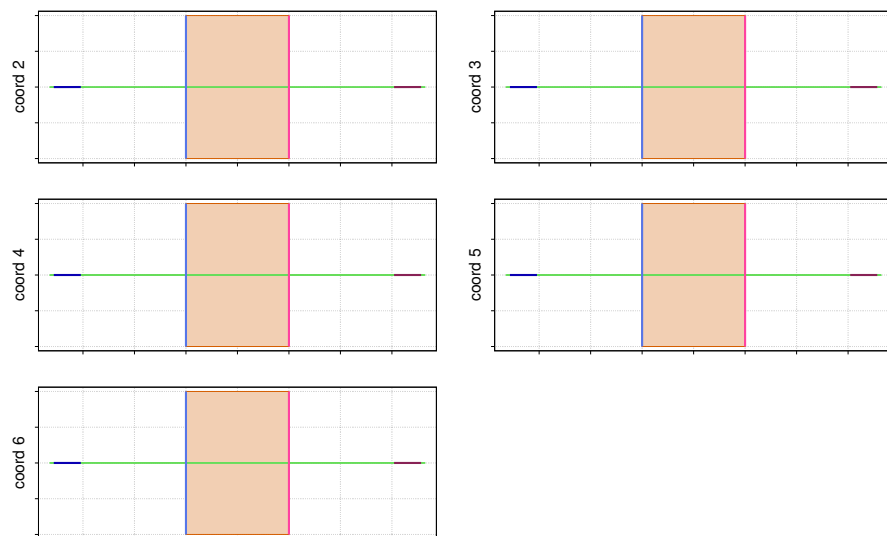


Figure 2: *Covering relation of the h -set N (orange) under the Poincaré map (green). The horizontal axis is the first, apparently unstable, coordinate. Blue and pink lines indicate the exit boundary N_c^- (lighter) and its image (darker).*

We note that this is just the preliminary result. The plan for the nearest future is to prove the following:

Conjecture 1. *There exists a homoclinic orbit to the periodic orbit given in Theorem 1.*

The insight into the dynamics of the ODE (4) will then be used to guide future research into proving similar results for the full DDE (3).

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