

A homoclinic connection to a vertical Lyapunov orbit in the Sun-Jupiter system: a computer-assisted proof

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Introduction

Homoclinic and heteroclinic orbits to equilibria and periodic solutions arise when their unstable and stable manifolds intersect. They play an important role in spacecraft mission design [7]. For example, the heteroclinic connection between L_1 and L_2 was used in the Genesis mission [3] and ARTEMIS [2].

We present a computer-assisted proof of a non-transversal homoclinic connection to a vertical Lyapunov orbit in the Circular Restricted Three-Body Problem (CR3BP) for the Sun-Jupiter system with mass parameter $\mu = 9.5388114032796904 \cdot 10^{-4}$. The proof exploits the reversing symmetry of the system and uses a topological argument based on the Brouwer degree. All rigorous computations are performed in the CAPD library [4]. In particular, CAPD is used to perform a rigorous enclosure of the Poincaré map [5].

Vertical Lyapunov orbits

The equations of motion of the CR3BP with mass parameter μ admit a first integral, the Jacobi constant C_μ , and two symmetries: a reversing

symmetry R and a symmetry S of the form

$$\begin{aligned} R(x, y, z, \dot{x}, \dot{y}, \dot{z}) &= (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}) \\ S(x, y, z, \dot{x}, \dot{y}, \dot{z}) &= (x, y, -z, \dot{x}, \dot{y}, -\dot{z}). \end{aligned}$$

Vertical Lyapunov orbits near libration points $\{L_i\}_{i=1,2,3}$ are double symmetric with respect to S and R . Using these symmetries, finding such an orbit at the chosen energy value e of C_μ reduces to finding an initial condition of the form $v_e = (x_e, 0, z_e, 0, \dot{y}_e, 0) \in \text{Fix}(R)$, which is a solution to the following parameterized equation

$$\begin{cases} \pi_{(y,\dot{x})}\mathcal{P}_z(x, 0, z, 0, \dot{y}, 0) = 0 \\ C_\mu(x, 0, z, 0, \dot{y}, 0) - e = 0. \end{cases}$$

The existence of a smooth family $e \mapsto v_e$ over an explicit energy interval E is verified using the Interval Newton Method [6] for parameterized equations. Moreover, $v_e \in \text{Fix}(R)$ is a fixed point of the fourth iterate of the Poincaré map on $\Pi_y = \{y = 0\}$:

$$F(v_e) = \mathcal{P}_y^4(v_e) = v_e.$$

Homoclinic connection

Since the system is reversible and $v_e \in \text{Fix}(R)$, any point $w \in W^u(v_e, F) \cap \text{Fix}(R)$ gives rise to a homoclinic connection to the vertical orbit. This reduces to finding a solution of

$$\pi_{(\dot{x}, \dot{z})}\mathcal{P}_y\mathcal{P}_x(w) = 0 \quad w \in W^u(v_e, F). \quad (1)$$

The spectrum of $A = DF(v_e)$ is of the form $\{\lambda, \lambda^{-1}, a \pm bi, 1\}$ with $\lambda > 1$ and $a^2 + b^2 = 1$. Hence the unstable manifold $W^u(v_e, F)$ is one-dimensional. We work with the Poincaré map on the section $\Pi_y = \{y = 0\}$, so the dynamics is described by a 5-dimensional map. After removing the neutral direction via the Jacobi constant and a change of basis L , diagonalizing A , the equation (1) becomes a system of two equations with two variables (τ, e) , where τ parameterizes $W^u(v_e, F)$

$$G(\tau, e) = \pi_{(\dot{x}, \dot{z})}\mathcal{P}_y\mathcal{P}_x(\tau, w_e^u(\tau)) = 0.$$

Rigorous estimate of unstable manifold and Brouwer degree argument

The local unstable manifold $W^u(v_e, F)$ is rigorously estimated using cone conditions [1]. In the coordinate system induced by L , we verify that the map F satisfies cone conditions for two quadratic forms Q_h, Q_v over an explicit neighborhood of the fixed point via interval enclosure of its derivative. This gives a Lipschitz function $w_e^u(\tau)$ whose graph is locally $W^u(v_e, F)$, together with explicit bounds used to estimate G .

To prove the existence of a zero of G we use the Brouwer degree. Given a numerical approximation $\tilde{\zeta} = (\tilde{\tau}, \tilde{e})$ and an explicit box $U \subset \mathbb{R}^2$ around $\tilde{\zeta}$, we form the homotopy

$$H(t, \zeta) = tB(\zeta - \tilde{\zeta}) + (1 - t)G(\zeta), \quad t \in [0, 1],$$

where B is a linear isomorphism. If $0 \notin H(t, \partial U)$ for every $t \in [0, 1]$, then $\deg(G, U, 0) \neq 0$ and G has a zero in U .

Main result

Theorem 1. *Let*

$$\begin{aligned} \mu &= 9.5388114032796904 \cdot 10^{-4} \\ \tilde{e} &= 3.0268216642156945825 \end{aligned}$$

and

$$E = \tilde{e} + 10^{-8} \cdot [-1, 1].$$

In the CR3BP with fixed parameter μ , there exists $e \in E$ such that $C_\mu^{-1}(\{e\})$ contains a vertical Lyapunov orbit near L_1 admitting a homoclinic connection.

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